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Computer Assignment 10

Markov Chains and Stochastic Matrices

(Atlast manual, beginning on page 154)

Chapter 7, section 2, problem 15a: Markov Chain for Voting Trends

A **stoachastic process** is any sequence of experiments in which the outcome at any stage depends on chance. A **Markov process** is a stochastic process with the following properties:

(i) The set of possible outcomes or states is finite.

(ii) The probability of the next outcome depends only on the previous outcome.

(iii) The probabilities are constant over time.

A square matrix is said to be **stoachastic** if each of its columns is a **probability vector**, i.e. adds up to 1.

Matlab Input:

```
A = [0.90 0.20;0.10 0.80]
x0 = [0.50 0.50]'
x1 = A * x0, x2 = A * x1, x3 = A * x2, A2 = A^2 * x0, A3 = A^3 * x0
```

Matlab Output:

```
A =
    0.9000    0.2000
    0.1000    0.8000
x0 =
    0.5000
    0.5000
x1 =
    0.5500
    0.4500
x2 =
    0.5850
    0.4150
x3 =
    0.6095
    0.3905
A2 =
    0.5850
    0.4150
A3 =
    0.6095
    0.3905
```

$\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are ratios of Democratic votes to Republican votes in the three elections following the one denoted by $\mathbf{x}_0$. It is found that $A^2\mathbf{x}_0 = \mathbf{x}_2$ and $A^3\mathbf{x}_0 = \mathbf{x}_3$.

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Chapter 7, section 2, problem 15b: Markov Chain for Voting Trends

Matlab Input:

```
A = [0.90 0.20;0.10 0.80]
x0 = [0.50 0.50]'
x10=A^10*x0, x15=A^15*x0, x20=A^20*x0, x25=A^25*x0, x30=A^30*x0
```

Matlab Output:

```
A =
    0.9000    0.2000
    0.1000    0.8000
x0 =
    0.5000
    0.5000
x10 =
    0.6620
    0.3380
x15 =
    0.6659
    0.3341
x20 =
    0.6665
    0.3335
x25 =
    0.6666
    0.3334
x30 =
    0.6667
    0.3333
```

The sequence of state vectors appears to be converging to $\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$.

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Chapter 7, section 2, problem 15c: Markov Chain for Voting Trends

Matlab Input:

```
A=[0.90 0.20;0.10 0.80]
x0 = [0.25 0.75]'
x10=A^10*x0,
x15=A^15*x0,
x20=A^20*x0,
x25=A^25*x0,
x30=A^30*x0
```

Matlab Output:

```
A =
    0.9000    0.2000
    0.1000    0.8000
x0 =
    0.2500
    0.7500
x10 =
    0.6549
    0.3451
x15 =
    0.6647
    0.3353
x20 =
    0.6663
    0.3337
x25 =
    0.6666
    0.3334
x30 =
    0.6667
    0.3333
```

Matlab Input:

```
A=[0.90 0.20;0.10 0.80]
x0 = [1.00 0.00]'
x10=A^10*x0,
x15=A^15*x0,
x20=A^20*x0,
x25=A^25*x0,
x30=A^30*x0
```

Matlab Output:

```
A =
    0.9000    0.2000
    0.1000    0.8000
x0 =
     1
     0
x10 =
    0.6761
    0.3239
x15 =
    0.6682
    0.3318
x20 =
    0.6669
    0.3331
x25 =
    0.6667
    0.3333
x30 =
    0.6667
    0.3333
```

Matlab Input:

```
A=[0.90 0.20;0.10 0.80]
x0 = [0.00 1.00]'
x10=A^10*x0,
x15=A^15*x0,
x20=A^20*x0,
x25=A^25*x0,
x30=A^30*x0
```

Matlab Output:

```
A =
    0.9000    0.2000
    0.1000    0.8000
x0 =
     0
     1
x10 =
    0.6478
    0.3522
x15 =
    0.6635
    0.3365
x20 =
    0.6661
    0.3339
x25 =
    0.6666
    0.3334
x30 =
    0.6667
    0.3333
```

These all converge to the same steady state vector, $\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$.

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Chapter 7, section 2, problem 15d: Markov Chain for Voting Trends

Matlab Input:

```
A = [0.90 0.20;0.10 0.80]
x = [2/3 1/3]'
b = A * x
Eigenvalues = eig(A), [V,D] = eig(A); Eigenvectors = V
```

Matlab Output:

```
A =
    0.9000    0.2000
    0.1000    0.8000
x =
    0.6667
    0.3333
b =
    0.6667
    0.3333
Eigenvalues =
    1.0000
    0.7000
Eigenvectors =
    0.8944   -0.7071
    0.4472    0.7071
```

It is seen for the steady state vector **x**, that $A\mathbf{x} = \mathbf{x}$. It is also seen that the 2:1 ratio found in the steady state vector **x** is reflected in the eigenvector of the dominant eigenvalue.

Chapter 7, section 2, problem 15e: Markov Chain for Voting Trends

Matlab Input:

```
A = [0.90 0.20;0.10 0.80]
Eigenvalues = eig(A), [V,D] = eig(A); Eigenvectors = V
```

Matlab Output:

```
A =
    0.9000    0.2000
    0.1000    0.8000
Eigenvalues =
    1.0000
    0.7000
Eigenvectors =
    0.8944   -0.7071
    0.4472    0.7071
```

It is seen that the steady state vector is also a probability vector as its components add up to 1. There appears to be only one steady state vector possible.

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Chapter 7, section 2, problem 16: Stochastic Matrices

Matlab Input:

```
A = randstoc(2)
EigenvaluesA = eig(A), SumAmI = sum(A-eye(2))
B = randstoc(3)
EigenvaluesB = eig(B), SumBmI = sum(B-eye(3))
C = randstoc(4)
EigenvaluesC = eig(C), SumCmI = sum(C-eye(4))
D = randstoc(5)
EigenvaluesD = eig(D), SumDmI = sum(D-eye(5))
E = randstoc(6)
EigenvaluesE = eig(E), SumEmI = sum(E-eye(6))
```

Matlab Output:

```
A =
    0.8072    0.3025
    0.1928    0.6975
EigenvaluesA =
    1.0000
    0.5047
SumAmI =
  1.0e-016 *
    0.5551    0.5551
B =
    0.4129    0.0473    0.0281
    0.1847    0.2883    0.3951
    0.4024    0.6644    0.5768
EigenvaluesB =
    0.3764
    1.0000
   -0.0984
SumBmI =
  1.0e-015 *
         0    0.2220    0.1110
C =
    0.1560    0.0071    0.2066    0.1170
    0.4741    0.3491    0.1856    0.3879
    0.2137    0.2081    0.3751    0.4838
    0.1561    0.4356    0.2328    0.0113
EigenvaluesC =
   1.0000
   0.1010 + 0.1698i
   0.1010 - 0.1698i
  -0.3104
SumCmI =
  1.0e-015 *
         0   -0.1110   -0.1665    0.1110
D =
    0.2194    0.2384    0.1462    0.2322    0.2500
    0.1222    0.1693    0.2615    0.2305    0.1962
    0.2679    0.1054    0.0728    0.1603    0.2489
    0.1619    0.1075    0.3369    0.1341    0.2009
    0.2285    0.3793    0.1826    0.2429    0.1040
EigenvaluesD =
   1.0000
  -0.0663 + 0.1401i
  -0.0663 - 0.1401i
  -0.1254
  -0.0423
SumDmI =
  1.0e-015 *
    0.0555   -0.1110   -0.1943    0.0833         0
```

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```
E =
    0.0953    0.1707    0.1573    0.3012    0.0050    0.1541
    0.1122    0.1113    0.2013    0.0834    0.3805    0.0213
    0.1757    0.2112    0.2423    0.0776    0.0848    0.3247
    0.2392    0.1643    0.1323    0.2692    0.1272    0.1914
    0.1017    0.1337    0.2229    0.2266    0.2815    0.1391
    0.2758    0.2088    0.0438    0.0420    0.1211    0.1693
EigenvaluesE =
   1.0000
  -0.1144
   0.0220 + 0.1569i
   0.0220 - 0.1569i
   0.0561
   0.1833
SumEmI =
  1.0e-015 *
    0.1110   -0.1388   -0.0347   -0.0902    0.1804         0
```

(a) The sum of the row vectors *A* - *I* appears to be zero for all stochastic matrices.

(b) The dominant eigenvalue of a stochastic matrix is 1.

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Chapter 7, section 2, problem 17: Stochastic Matrices

Matlab Input:

```
A = [0.90 0.20;0.10 0.80]
x0 = [0.50 0.50]', x1 = A * x0, x2 = A * x1
Eigenvalues = eig(A), [V,D]=eig(A); v1=V(:,1), v2=V(:,2)
C0=V\x0, C1=V\x1, C2=V\x2
```

Matlab Output:

```
A =
    0.9000    0.2000
    0.1000    0.8000
x0 =
    0.5000
    0.5000
x1 =
    0.5500
    0.4500
x2 =
    0.5850
    0.4150
Eigenvalues =
    1.0000
    0.7000
v1 =
    0.8944
    0.4472
v2 =
   -0.7071
    0.7071
C0 =
    0.7454
    0.2357
C1 =
    0.7454
    0.1650
C2 =
    0.7454
    0.1155
```

$$x_0 = 0.7454v_1 + 0.2357v_2$$

$$x_1 = 0.7454v_1 + 0.1650v_2$$

$$x_2 = 0.7454v_1 + 0.1155v_2$$

I don't recognize the formula although it appears that c_1 is always 0.7454 and c_2 is reduced by a factor of 0.7 for each progressive state vector.

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Chapter 7, section 2, problem 18: A problem regarding the Speckled Phoenix

Matlab Input:

```
A = [0 0 0.4;0.2 0 0;0 0.7 0.9]
Eigenvalues = eig(A),
```

Matlab Output:

```
A =
         0         0    0.4000
    0.2000         0         0
         0    0.7000    0.9000
Eigenvalues =
  -0.0303 + 0.2395i
  -0.0303 - 0.2395i
   0.9607
```

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ABOUT THE MATLAB COMMANDS USED

---

apostrophe '   Matrix transpose.

Synopsis: `A'`

the linear algebraic transpose of A. For complex matrices, this involves the complex conjugate transpose.

**backslash **

Synopsis: `X = A\B`

X = A\B is the solution to the equation AX = B

---

colon :   Create vectors, etc.

Synopsis: `v = A(:,j)`

A(:,j) is the j-th column of A

---

eig   Eigenvalues.

Synopsis:
```
e = eig(A)
[V,D] = eig(A)
```

`E = eig(A)` is a vector containing the eigenvalues of a square matrix A.

`[V,D] = eig(A)` produces a diagonal matrix *D* of eigenvalues and a full matrix *V* whose columns are the corresponding eignvectors.

---

eye   Identity matrix.

Synopsis:
```
Y = eye(n)
Y = eye(size(A))
```

`Y = eye(n)` is the n-by-n identity matrix.

`Y = eye(size(A))` is the same size as A.

---

randstoc   Create a stochastic matrix (Atlast command)

Synopsis: `randstoc(n)`

The command `A=randstoc(n)` will generate a random $n \times n$ stochastic matrix (each column adds up to 1).
