FOURIER SERIES
Continuous-Time Signal Analysis

The **Fourier series** is a method of expressing most periodic, time-domain functions in the frequency domain. The frequency domain representation appears graphically as a series of spikes occurring at the fundamental frequency (determined by the period of the original function) and its harmonics. The magnitudes of these spikes are the **Fourier coefficients**. This series of components are called the **wave spectrum**.

THE EXPONENTIAL FOURIER SERIES

$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

where:

D is the amplitude or coefficient
n is the harmonic
ω_0 is the radian frequency [rad/s]
$\omega_0/2\pi$ is the frequency [hertz]
$T_0 = 2\pi/\omega_0$ is the period [sec.]

$$D_n = \frac{1}{T_0} \int_{T_0} f(t) e^{-jn\omega_0 t} dt \qquad D_0 = C_0 = a_0$$

$$\omega_0 = \frac{2\pi}{T_0}$$

The exponential form carries no more and no less information than the other forms but is preferred because it requires less calculation to determine and also involves simpler calculations in its use. One drawback is that the exponential form is not as easy to visualize as trigonometric forms.

THE TRIGONOMETRIC FOURIER SERIES

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

where:

a is the amplitude or coefficient
n is the harmonic
θ is the phase
ω_0 is the radian frequency [rad/s]
$\omega_0/2\pi$ is the frequency [hertz]
$T_0 = 2\pi/\omega_0$ is the period [sec.]

$$a_0 = \frac{1}{T_0} \int_{T_0} f(t)\, dt \qquad \omega_0 = \frac{2\pi}{T_0}$$

$$a_n = \frac{2}{T_0} \int_{T_0} f(t) \cos n\omega_0 t\, dt = C_n \cos\theta_n$$

$$b_n = \frac{2}{T_0} \int_{T_0} f(t) \sin n\omega_0 t\, dt = -C_n \sin\theta_n$$

The Fourier series of an even periodic function will consist of cosine terms only and an odd periodic function will consist of sine terms only.

THE COMPACT TRIGONOMETRIC FOURIER SERIES

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

where:

C is the amplitude or coefficient n is the harmonic θ is the phase ω_0 is the radian frequency [rad/s] $\omega_0/2\pi$ is the frequency [hertz] $T_0 = 2\pi/\omega_0$ is the period [sec.]	$C_n = \sqrt{a_n^2 + b_n^2}$ $\qquad C_0 = a_0$ $a_n, b_n:$ *see above* $\theta_n = \tan^{-1}\left(\dfrac{-b_n}{a_n}\right) \qquad \omega_0 = \dfrac{2\pi}{T_0}$

THE DIRICHLET CONDITIONS

WEAK DIRICHLET CONDITION - For the Fourier series to exist, the function $f(t)$ must be absolutely integrable over one period so that coefficients a_0, a_n, and b_n are finite. This guarantees the existence of a Fourier series but the series may not converge at every point.

STRONG DIRICHLET CONDITIONS - For a convergent Fourier series, we must meet the weak Dirichlet condition and $f(t)$ must have only a finite number of maxima and minima in one period. It is permissible to have a finite number of finite discontinuities in one period.

FREQUENCY SPECTRA

AMPLITUDE SPECTRUM - The plot of amplitude C_n versus the (radian) frequency ω (use the function $C_n = \sqrt{a_n^2 + b_n^2}$. A point is plotted for each harmonic ($n = 0, 1, 2, 3, \cdot \cdot \cdot$) so you need to know the relationship between n and ω. Although the formula for C_n suggests that the value is always positive, it is sometimes *convenient* to let the values go negative to make the graph smooth.

PHASE SPECTRUM - The plot of the phase θ_n versus the (radian) frequency ω.

These two plots convey all of the information that the plot of $f(t)$ as a function of t does. The frequency spectra of a signal constitute the **frequency-domain** description of $f(t)$ whereas $f(t)$ as a function of t is the **time-domain** description.

PERIODIC FUNCTIONS

For a function to be **periodic**, all frequencies in its spectrum must be **harmonically related**. This means that the ratio of any two frequencies must be a rational number. For example in the function:

$$f(t) = 5\cos(2t + \theta) + \sin 3t + 4\sin(\pi t + \phi)$$

we have three frequencies. The ratio of the first to the second is 2/3 so that they are harmonically related. The ratio of the first to the third is $2/\pi$ so they are not harmonically related and the function is therefore not periodic.

BANDWIDTH

The difference (in radians) of the highest and lowest frequencies in the **phase spectrum** is the **bandwidth**.