CHAPTER 16
Multiple Integration

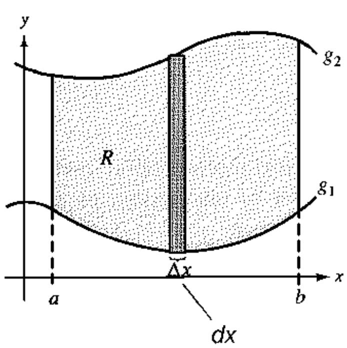
When evaluating the integral of a multivariable function, hold constant the variables not being integrated.

An **iterated integral** has two integration symbols. The inside one is evaluated first. It may have variables defining the interval, but the outside one may not. And if the **inside** limits of integration use the variable ***x***, for example, the **outside** integral must have a ***dx*** on it. By this process, the variables of integration are removed as the problem is worked. The inside limits define the area more precisely while the outer limits confine the extent of the area in more general terms. It will be necessary to graph the area to visualize this.

Note that the difference between finding the **area of a region** and the **volume of a region** using a double integral is that the double integral for the volume of a region contains a function between the symbols of integration and $dy\ dx$ while this area is empty when only the area is being found. The aforementioned function represents the top or cap of the solid while the limits represent the region in the xy plane. When graphing the problem, you are concerned with graphing the limits, not the function. Understanding this paragraph can save a lot of pain in this chapter.

Area of a Region in the Plane: 16.1 p934

1. If R is defined by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, where $g1$ and $g2$ are continuous on [a, b], then the area of r is given by



Vertically Simple:

$$A = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\ dx$$

2. If R is defined by $c \le y \le d$ and $h_1(y) \le x \le h_2(y)$, where h_1 and h_2 are continuous on [c, d], then the area of r is given by

Horizontally Simple: $$A = \int_c^d \int_{h_1(y)}^{h_2(y)} dx\ dy$$

A **double integral** is used to find the volume of a region and may be evaluated as an *iterated integral*:

Volume of a Region:

$$V = \int_a^b \int_c^d f(x,\ y)\ dy\ dx$$

where [a, b] are the limits of x (determined to be x by the position of dx in the expression),

where [c, d] are the limits of y because of the position of dy in the expression,

where $f(x,\ y)$ represents the cap or upper limits of the region and the lower limit is $z = 0$.

The inner limits tend to more precisely define the area, i.e. a variable or a function, while the outer limits are just constants.

Fubini's Theorem: 16.2 p942 Let f be continuous on a plane region R.

1. if R is defined by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, where g_1 and g_2 are continuous on [a, b], then
$$\int_R \int f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\ dx$$

2. If R is defined by $c \le y \le d$ and $h_1(y) \le x \le h_2(y)$, where h_1 and h_2 are continuous on [c, d], then
$$\int_R \int f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} dx\ dy$$

Polar form of Fubini's Theorem: 16.3 p948 Let f be continuous on a plane region R.

1. If R is defined by $\theta_1 \le \theta \le \theta_2$ and $g_1(\theta) \le r \le g_2(\theta)$, where g_1 and g_2 are continuous on [θ_1, θ_2], then
$$\int_R \int f(r,\theta)\,dA = \int_{\theta_1}^{\theta_2} \int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta) r\,dr\ d\theta$$

2. If R is defined by $r_1 \le r \le r_2$ and $h_1(r) \le \theta \le h_2(r)$, where h_1 and h_2 are continuous on [r_1, r_2], then
$$\int_R \int f(r,\theta)\,dA = \int_{r_1}^{r_2} \int_{h_1(r)}^{h_2(r)} f(r,\theta) r\,d\theta\ dr$$

Note than an r multiplier is added to the function.

Changing variables to polar form: 16.3 p950

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$r^2 = x^2 + y^2 \qquad dA = r\,dr\,d\theta$$

$$\theta = \arctan\frac{y}{x}$$

$$\int_R\int f(x,y)\,dA = \int_{\theta_1}^{\theta_2}\int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)r\,dr\ d\theta$$

The significance of the above equation is important. Note the appearance of the extra r in the right-hand side. This must be added when changing to polar form.

Mass of a Planar Lamina of Variable Density: 16.4 p954 If ρ is a continuous density function on the lamina corresponding to a plane region R, then the mass m of the lamina is given by

$$m = \int_R\int \rho(x,y)\,dA$$

Moments and Center of Mass of a Variable Density Planar Lamina: 16.4 p956 If ρ is a continuous density function on the lamina corresponding to a plane region R, then the **moments of mass** with respect to the x- and y-axes are

$$m_x = \int_R\int y\rho(x,y)\,dA \qquad \text{and} \qquad m_y = \int_R\int x\rho(x,y)\,dA$$

The **center of mass** is: $(\bar{x},\bar{y}) = \left(\dfrac{M_y}{m}, \dfrac{M_x}{m}\right)$

If R represents a simple plane region rather than a lamina, the point $(\bar{x},\bar{y})$ is called the **centroid** of the region.

The moments M_x and M_y used in the above equation are sometimes referred to as the **first moments** about the x- and y-axes. The moment is the product of a mass times a distance. 16.4 p957

$M_x = \int_R\int (y)\rho(x,y)\,dA$ — (y) is the distance to the x-axis; $\rho(x,y)\,dA$ is the mass

$M_y = \int_R\int (x)\rho(x,y)\,dA$ — (x) is the distance to the y-axis; $\rho(x,y)\,dA$ is the mass

The **second moment**, or the **moment of inertia** of a lamina about a line is a measure of the tendency of matter to resist a change in rotational motion, and is denoted by I_x and I_y. 16.4 p957

$I_x = \int_R\int (y^2)\rho(x,y)\,dA$ — (y) is the distance to the x-axis; $\rho(x,y)\,dA$ is the mass

$I_y = \int_R\int (x^2)\rho(x,y)\,dA$ — (x) is the distance to the y-axis; $\rho(x,y)\,dA$ is the mass

The sum of the moments I_x and I_y is called the **polar moment of inertia** and is denoted by $I_0 = I_x + I_y$.

Radius of Gyration: $\bar{\bar{r}} = \sqrt{\dfrac{I}{m}}$

Radius of gyration about the ***x*-axis**: $\bar{\bar{y}} = \sqrt{\dfrac{I_x}{m}}$

Radius of gyration about the ***y*-axis**: $\bar{\bar{x}} = \sqrt{\dfrac{I_y}{m}}$

Surface Area: 16.5 p961 If f and its first partial derivatives are continuous on the closed region R in the xy-plane, then the **area of the surface** $z = f(x, y)$ over R is given by

$$\text{surface area} = \int_R\int dS = \int_R\int \sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2}\,dA$$

A **triple integral** is used to find the volume of a region and may be evaluated by *iterated integrals*: 16.6 p968

$$\iiint_Q f(x,y,z)dV = \int_a^a\int_{h_1(x)}^{h_2(x)}\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)dz\,dy\,dx$$

However, the triple integral above is **not** a calculation of volume. When a triple integral is used to calculate volume, there is no function at the center of the expression as there is with a double integral. The boundaries of the solid region are defined by the limits, with the inner limit being the most specific.

To **evaluate a triple iterated integral** in the order dz, dy, dx, we hold **both** x and y constant for the innermost integration, and then hold x constant for the second integration.

The Cylindrical Coordinate System: 16.7 p976 In a **cylindrical coordinate system**, a point P in space is represented by an ordered triple (r, θ, z).

1. (r,) is a polar representation of the projection of P in the *xy*-plane.
2. z is the directed distance from (r,) to P.

Cylindrical to rectangular conversion:

$$x = r\cos\theta,\ \ y = r\sin\theta,\ \ z = z$$

Rectangular to cylindrical conversion:

$$r^2 = x^2 + y^2,\ \ \tan\theta = \frac{y}{x},\ \ z = z$$

The Spherical Coordinate System: 16.7 p980 In a **spherical coordinate system**, a point P in space is represented by an ordered triple (ρ, θ, ϕ). "(row, theta, fee)"

1. ρ is the distance between P and the origin, $r \geq 0$.
2. θ is the same angle used in cylindrical coordinates, $0 \leq \theta \leq 2\pi$.
3. ϕ is the angle between the positive z-axis and the line segment OP, $0 \leq \phi \leq \pi$.

Spherical to rectangular conversion:

$$x = \rho\sin\phi\cos\theta,\quad y = \rho\sin\phi\sin\theta,\quad z = \rho\cos\phi$$

Rectangular to spherical conversion:

$$\rho^2 = x^2 + y^2 + z^2,\quad \tan\theta = \frac{y}{x},\quad \phi = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

Note that when converting an integral in rectangular coordinates to an integral in spherical coordinates, $r^2 \sin f$ is added as a multiplier to the interior of the expression.

The Jacobian: 16.9 p991 If $x = g(u, v)$ and $y = h(u, v)$, then the **Jacobian** of x and y with respect to u and v, denoted by $\partial(x, y)/\partial(u, v)$, is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

Change of Variables for Double Integrals: 16.9 p993 Let R and S be regions in the *xy*- and *uv*-planes that are related by the equations $x = g(u, v)$ and $y = h(u, v)$ such that each point in R is the image of a unique point in S. If f is continuous on R and g and h have continuous partial derivatives on S and $\partial(x, y)/\partial(u, v)$ is nonzero on S, then

$$\int_R\int f(x, y)\,dA = \int_S\int f(g(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Find the Volume of a Region over a parallelogram using a Change of Variables: Find the equations of the lines bounding the parallelogram and arrange so that x and y are on one side of the equation and a constant is on the other. Let u equal the variable side of the equation for one pair of parallel lines and v equal the variable side of the equation for the other pair of lines.

Using these two equations as a guide, set up a table to convert the 4 coordinates of the corners of the parallelogram to new coordinates in terms of u and v. From these, the new limits (constants) are determined.

Now solve the 2 equations for x and y. Make substitutions so that x and y are expressed in terms of u and v. Take the partial derivatives of this x and y to find the **Jacobian** $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$. Take the product of the upper left and lower right. Subtract from that the product of the lower left and upper right. The absolute value of the result is the Jacobian.

Form a new function by substituting the new expressions for x and y into the original function. Multiply that by the Jacobian and set up the result in a double integral using the new limits obtained from the (u, v) coordinates.

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