CHAPTER 13
Vectors and Curves in the Plane

A vector whose initial point is at the origin is said to be in **standard position**. A variable which represents a vector may be written $\vec{v}$ or **v**.

Definition of Component Form of a Vector in the Plane:
13.1 p728 If $\vec{v}$ is a vector in the plane whose initial point is the origin and whose terminal point is (v_1, v_2), then the component form of $\vec{v}$ is given by

$$\vec{v} = \langle v_1,\ v_2 \rangle$$

The coordinates v_1 and v_2 are called the **components of** $\vec{v}$. If both the initial point and the terminal point lie at the origin, then $\vec{v}$ is called the **zero vector** and is denoted by

$$0 = \langle 0, 0 \rangle$$

To convert Directed Line Segments to Component form or vice versa, we use the following procedures. 13.1 p728

1. If $P = p_1, p_2$) and $Q = (q_1, q_2)$, then the component form of the vector $\vec{v}$ represented by $P\vec{Q}$ is $\langle v_1, v_2 \rangle = \langle q_1 - p_1, q_2 - p_2 \rangle$. Moreover, the **<u>magnitude</u> or <u>norm</u>** (length) of $\vec{v}$ is given by
$\|\vec{v}\| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2} = \sqrt{v_1^{\ 2} + v_2^{\ 2}}$ length of a vector

The length of v is also called the **norm of** $\vec{v}$. If $\|\vec{v}\| = 1$, then $\vec{v}$ is called a **unit vector**.

2. If $\vec{v} = \langle v_1,\ v_2 \rangle$, then $\vec{v}$ can be represented by the directed line segment, in standard position, from $P = (0, 0)$ to $Q = (v_1, v_2)$.

Standard Unit Vectors: 13.1 p733

$$\vec{i} = \langle 1, 0 \rangle \qquad \vec{j} = \langle 0, 1 \rangle$$

We call $\vec{v} = v_1\mathbf{i} + v_2\mathbf{j}$ a **linear combination** of i and j. The scalars v_1 and v_2 are called the **horizontal** and **vertical components of** $\vec{v}$, respectively. In other words, the vector $\langle 2, 3 \rangle$ expressed in **component form** would be 2i + 3j.

Definition of Vector Addition and Scalar Multiplication:
13.1 p730 For vectors $\vec{u} = \langle u_1,\ u_2 \rangle$ and $\vec{v} = \langle v_1,\ v_2 \rangle$ and scalar k, we define the following operations.

1. The **vector sum** of $\vec{u}$ and $\vec{v}$ is the vector $\vec{u} + \vec{v} = \langle u_1 + v_1,\ u_2 + v_2 \rangle$
2. The **scalar multiple** of k and $\vec{u}$ is the vector $k\vec{u} = \langle ku_1,\ ku_2 \rangle$.
3. The **negative** of $\vec{v}$ is the vector $-\vec{v} = (-1)\vec{v} = \langle -v_1, -v_2 \rangle$.
4. The **difference** of $\vec{u}$ and $\vec{v}$ is $\vec{u} - \vec{v} = \vec{u} + (-\vec{v}) = \langle u_1 - v_1,\ u_2 - v_2 \rangle$

Properties of Vector Addition and Scalar Multiplication:
13.1 p731 Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in the plane, and let c and d be scalars.

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ commutative property
2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ associative property
3. $\vec{u} + 0 = \vec{u}$
4. $\vec{u} + (-\vec{u}) = 0$
5. $c(d\vec{u}) = (cd)\vec{u}$
6. $(c + d)\vec{u} = c\vec{u} + d\vec{u}$
7. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$ distributive property
8. $1(\vec{u}) = \vec{u},\quad 0(\vec{u}) = 0$ distributive property

Length of a Scalar Multiple: 13.1 p731 Let $\vec{v}$ be a vector and c be a scalar. Then
$\|c\vec{v}\| = |c|\ \|\vec{v}\|$ where **|**c**|** is the absolute value of c.

<u>Unit vector</u> <u>in the direction of $\vec{v}$</u>: 13.1 p732 If $\vec{v}$ is a nonzero vector in the plane, then the vector

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{l}{\|\vec{v}\|}\vec{v}$$

has length l and the same direction as $\vec{v}$.

Triangle Inequality: 13.1 p733 If $\vec{u}$ and $\vec{v}$ are vectors in the plane, then

$$\|\vec{u} + \vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|.$$

Determining a Vector as a function of an angle: 13.1 p734 If $\vec{v}$ is a *unit vector* with angle θ with respect to the positive x-axis, then $\vec{u} = \langle \cos\theta, \sin\theta \rangle = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$

Dot Product: 13.2 p738 The **dot product** of $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$ is $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$. The dot product is a product of vector values; the i and j are dropped from the result.

Properties of the Dot Product: 13.2 p738 If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in the plane and c is a scalar, then the following properties are true.

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ commutative property
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ distributive property
3. $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$
4. $0 \cdot \vec{v} = 0$
5. $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

Angle between two vectors: 13.2 p739 If θ is the angle between two nonzero vectors $\vec{u}$ and $\vec{v}$, then

$$\cos\theta = \frac{\vec{u} + \vec{v}}{\|\vec{u}\|\|\vec{v}\|} \quad \text{and} \quad \vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$$

Definition of Orthogonal Vectors: 13.2 p740 The vectors $\vec{u}$ and $\vec{v}$ are **orthogonal** if $\vec{u} \cdot \vec{v} = 0$. We consider the zero vector to be orthogonal to every vector $\vec{u}$ since $0 \cdot \vec{u} = 0$.

Two nonzero vectors are **parallel** is they have the same or opposite directions.

Projection and Vector Components: 13.2 p742 Let $\vec{u}$ and $\vec{v}$ be nonzero vectors with $\vec{u} = \vec{w}_1 + \vec{w}_2$, where $\vec{w}_1$ is parallel to $\vec{v}$ and $\vec{w}_2$ is orthogonal to $\vec{v}$.

$\vec{w}_1$ is the **projection of** $\vec{u}$ **onto** $\vec{v}$ or the vector component of $\vec{u}$ along $\vec{v}$, and is denoted by $\vec{w}_1 = \text{proj}_v \vec{u}$.

$\vec{w}_2 = \vec{u} - \vec{w}_1$ and is called the **vector component of** $\vec{u}$ **orthogonal to** $\vec{v}$.

We can find the vector component $\vec{w}_2$ once we have found the projection of $\vec{u}$ onto $\vec{v}$.

Finding Projection: 13.2 p742

$$\text{proj}_v \vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \right) \vec{v}$$

Definition of Work: 13.2 p744 The work W done by a constant force $\vec{F}$ as its point of application moves along the vector $\vec{PQ}$ is given by either of the following:

1) $W = \|\text{proj}_{\vec{PQ}} \vec{F}\| \|\vec{PQ}\|$ Projection form

2) $W = \vec{F} \cdot \vec{PQ}$ Dot product form

Vector-Valued Function: 13.3 p746 A function of the form:

$$\vec{r}(t) = f(t)i + g(t)j$$

is called a **vector-valued function**, where the component functions, f and g, are real-valued functions of the parameter t.

The **domain** of the vector-valued function $\vec{r}$ is the intersection of the domains of f and g.

The **limit** of the vector-valued function $\vec{r}$ is

$$\lim_{t \to a} \vec{r}(t) = \left[\lim_{t \to a} f(t)\right]i + \left[\lim_{t \to a} g(t)\right]j$$

provided that f and g have limits as $t \to a$.

The vector-valued function $\vec{r}$ is **continuous** at the point given by $t = a$ if the limit exists as $t \to a$ and

$$\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$$

A vector-valued function is **continuous** on an interval if it is continuous at every point in the interval.

The vector-valued function $\vec{r}(t) = f(t)i + g(t)j$ is **smooth** on an open interval I if f' and g' are continuous on I and $\vec{r}\,'(t) \neq 0$ for any value of t in the interval I.

The **Derivative** of a Vector-Valued Function: 13.3 p750

$$\vec{r}\,'(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t}$$

for all t for which the limit exists.

Differentiation of Vector-Valued Functions: 13.3 p750 If $\vec{r}(t) = f(t)i + g(t)j$ where f and g are differentiable functions of t, then $\vec{r}\,'(t) = f'(t)i + g'(t)j$.

Properties of the derivative of Vector-Valued Functions: 13.3 p752

1) $D_t[c\,\vec{r}(t)] = c\,\vec{r}\,'(t)$
2) $D_t[\vec{r}(t) \pm \vec{u}(t)] = \vec{r}\,'(t) \pm \vec{u}\,'(t)$
3) $D_t[f(t)\vec{r}(t)] = f(t)\vec{r}\,'(t) + f'(t)\vec{r}(t)$
4) $D_t[\vec{r}(t) \cdot \vec{u}(t)] = \vec{r}(t) \cdot \vec{u}\,'(t) + \vec{r}\,'(t) \cdot \vec{u}(t)$
5) $D_t[\vec{r}(f(t))] = \vec{r}\,'(f(t))f'(t)$
6) If $\vec{r}(t) \cdot \vec{r}(t) = c$, then $\vec{r}(t) \cdot \vec{r}\,'(t) = 0$.

Integration of a Vector-Valued Function: 13.3 p753 If $\vec{r}(t) = f(t)i + g(t)j$ where f and g are continuous on [a, b], then the **indefinite integral** of $\vec{r}$ is

$$\int \vec{r}(t)\,dt = \left[\int f(t)\,dt\right]i + \left[\int g(t)\,dt\right]j$$

and its **definite integral** over the interval $a \le t \le b$ is

$$\int_a^b \vec{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]i + \left[\int_a^b g(t)\,dt\right]j$$

Velocity and Acceleration: 13.4 p757 For the function $\vec{r}(t) = x(t)i + y(t)j$,

velocity = $\vec{v}(t) = \vec{r}\,'(t) = x'(t)i + y'(t)j$

acceleration = $\vec{a}(t) = \vec{r}\,''(t) = x''(t)i + y''(t)j = a_{\vec{T}}\vec{T}(t) + a_{\vec{N}}\vec{N}(t)$

speed = $\|\vec{v}(t)\| = \|\vec{r}\,'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2} = \dfrac{ds}{dt}$

Circular Function: 13.4 p758 The rectangular equation for a circle is $x^2 + y^2 = r^2$. The vector-valued function for a circular path of counterclockwise motion is:

$$\vec{c}(t) = r\cos\frac{[\text{spd}]t}{r}i + r\sin\frac{[\text{spd}]t}{r}j$$

where r = radius
t = time
[spd] = speed

When the motion is at a constant speed, the velocity vector is tangent to the circle and the acceleration vector is directed toward the center.

Gravitational Constant: 13.4 p760 g = 32 ft./sec./sec.
g = 9.81 meters/sec./sec.

Position Function for a Projectile: 13.4 p761 The path of a projectile launched from an initial height h with initial speed v_0 and angle of elevation q is described by the vector function

$$\vec{r}(t) = (v_0\cos\theta)t\,i + \left[h + (v_0\sin\theta)t - \frac{1}{2}gt^2\right]j$$

Unit Tangent Vector: 13.5 p764 Let C be a smooth curve represented by $\vec{r}$ on an open interval I. If $\vec{r}\,'(t) \ne 0$, then the unit tangent vector $\vec{T}(t)$ at t is defined to be

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$$

Normal means **perpendicular**.

Principal Unit Normal Vector: 13.5 p765 Let C be a smooth curve represented by $\vec{r}$ on an open interval I. If $\vec{T}'(t) \ne 0$, then the principal unit normal vector at t is defined to be

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$

The **normal vector** $\vec{N}(t)$ can be difficult to calculate but if $\vec{T}(t) = x(t)i + y(t)j$, and there is zero acceleration, then $\vec{N}(t)$ is either

$y(t)i - x(t)j$ [clockwise curve]

or $-y(t)i + x(t)j$ [counterclockwise curve]

The correct choice is the vector pointing to the inside of the curve of $\vec{r}(t)$. In other words, $\vec{N}(t)$ is at a right angle to $\vec{T}(t)$ which yields the two possibilities.

Tangential and Normal Components of Acceleration: 13.5 p768 Let $\vec{r}(t)$ be the position vector for a smooth curve C, and let the acceleration vector be represented by

$$\vec{a}(t) = a_{\vec{T}}\vec{T} + a_{\vec{N}}\vec{N} = (\vec{a}\cdot\vec{T})\vec{T} + (\vec{a}\cdot\vec{N})\vec{N}$$

We call $a_{\vec{T}} = \vec{a}\cdot\vec{T} = \dfrac{d^2s}{dt^2}$ the **tangential component of acceleration**, and we call $a_{\vec{N}} = \vec{a}\cdot\vec{N} = K\left(\dfrac{ds}{dt}\right)^2$ the **normal component of acceleration**.

Note that calculations for $a_{\vec{T}}$ and $a_{\vec{N}}$ involve points rather than functions so that the ***i*** and ***j*** do not appear in the result. Or at least I think that's what's going on.

Arc Length of a Plane Curve: 13.6 p772 If C is a smooth curve given by $\vec{r}(t) = x(t)i + y(t)j$ on an interval [a, b], then the arc length of C on the interval is

$$s = \int_a^b \|\vec{r}\,'(t)\|\,dt$$

Arc Length Function: 13.6 p722 Let C be a smooth curve given by $\vec{r}(t)$ defined on the closed interval [a, b]. For $a \le t \le b$, the **arc length function** is given by

$$s(t) = \int_a^t \|\vec{r}\,'(\tau)\|\,d\tau$$

t is the lowercase Greek letter tau.

Derivative of the Arc Length Function: 13.6 p773 Let C be a smooth curve given by $\vec{r}(t)$ defined on the closed interval [a, b]. The derivative of the arc length function s for this curve is given by

$$\frac{ds}{dt} = \|\vec{r}'(t)\|$$

In differential form, we can write $ds = \|\vec{r}'(t)\| dt$.

Arc Length Parameter: 13.6 p774 If C is a smooth curve given by $\vec{r}(s) = x(s)i + y(s)j$, where s is the arc length parameter, then

$$\|\vec{r}'(s)\| = 1$$

Moreover, if t is *any* parameter for the vector-valued function $\vec{r}$ such that $\|\vec{r}'(t)\| = 1$, then t must be the **arc length parameter**.

Curvature: 13.6 p774 Let C be a smooth curve given by $\vec{r}(s)$, where s is the **arc length parameter**. The curvature at s is given by

$$K = \left\|\frac{d\vec{T}}{ds}\right\| = \|\vec{T}'(s)\| = \|\vec{r}''(s)\|$$

If C is a smooth curve given by $\vec{r}(t)$, then the curvature of C at t is given by

$$K = \left\|\frac{\vec{T}'(t)}{\vec{r}'(t)}\right\|$$

If C is a curve with curvature K at point P, a circle passing through the point P with radius r = 1/K is called the **circle of curvature** if the circle lies on the concave side of the curve and shares a common tangent line with the curve at the point P. We call r the **radius of curvature** at P, and the center of the circle is called the **center of curvature**.

In rectangular coordinates, if C is the graph of a twice differentiable function given by $y = f(x)$, then the curvature at the point (x, y) is given by

$$\boxed{K = \frac{|y''|}{[1+(y')^2]^{3/2}}}$$

For a function given by $x = x(t)$, $y = y(t)$, then the curvature at the point (x, y) is given by

$$\boxed{K = \frac{|x'y'' - y'x''|}{[(x')^2 + (y')^2]^{3/2}}}$$

Acceleration, Speed, and Curvature: 13.6 p778 If $\vec{r}(t)$ is the position vector for a smooth curve C, then the acceleration vector is given by

$$\vec{a}(t) = \frac{d^2s}{dt^2}\vec{T} + K\left(\frac{ds}{dt}\right)^2 \vec{N}$$

where K is the curvature of C and ds/dt is the speed.

Force: 13.6 p778

$$\vec{F} = m\vec{a} = m\left(\frac{d^2s}{dt^2}\right)\vec{T} + mK\left(\frac{ds}{dt}\right)^2 \vec{N}$$

m = mass
K = curvature