PLANE CURVES, PARAMETRIC EQUATIONS, POLAR COORDINATES
Chapter 12

Definition of a **Plane Curve**: 12.1 p679 If f and g are continuous functions of t on an interval I, then the set of ordered pairs $(f(t), g(t))$ is called a plane curve C. The equations $x = f(t)$ and $y = g(t)$ are called **parametric equations** for C, and t is called the **parameter**. A parameter is a third variable that generates (x, y) or (r, θ) coordinates.

To find a pair of **parametric equations** to represent a function, find dy/dx and set that equal to x for the first equation. Substitute the value of y' for x in the function to obtain the second equation. 12.1 p683

To eliminate the parameter, solve for the parameter in one equation and substitute into the other. Where trigonometry is involved, solve for *sin* and *cos* and use identities.

The Orientation of the graph of Parametric Equations is the direction of *movement* along the graph relative to the order of the values of the parameter.

A curve C represented by $x = f(t)$ and $y = g(t)$ on an interval I is called **smooth** if f' and g' are continuous on I and not simultaneously zero, except possibly at the endpoints of I. The curve C is called **piecewise smooth** if it is smooth on each subinterval of some partition of I. 12.1 p684

Finding the **Derivative** of Parametric Equations: 12.2 p687 If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$, then the **slope** of C at (x, y) is

$$\frac{dy}{dx} = \frac{dy\ /\ dt}{dx\ /\ dt}, \quad \frac{dx}{dt} \neq 0 \qquad \text{First Derivative}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx\ /\ dt} \qquad \text{Second Derivative}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{\frac{d}{dt}\left(\frac{d^2y}{dx^2}\right)}{dx\ /\ dt} \qquad \text{Third Derivative}$$

Arc Length: 12.1 p690; 6.4 If a smooth curve C is given by $x = f(t)$ and $y = g(t)$ such that C does not intersect itself on the interval $a \geq t \geq b$ (except possibly at the endpoints), then the arc length of C over the interval is given by

$$s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\ dt$$

Cycloid - The figure formed by the path of a point on a circle as the circle is rolled along a line. 12.1 p683

Prolate cycloid - The figure formed by the path of a point outside a circle as the circle is rolled along a line. The path crosses itself once per revolution. 12.2 p689

Epicycloid - The figure formed by the path of a point on a circle as the circle is rolled along the circumference of another circle. 12.2 p690

Area of a Surface of Revolution: 12.2 p692 If a smooth curve C given by $x = f(t)$ and $y = g(t)$ does not cross itself on an interval $a \geq t \geq b$, then the area S of the surface of revolution formed by revolving C about the coordinate axes is given by the following.

Revolution about the x-axis:

$$s = 2\pi\int_a^b g(t)\sqrt{[f'(t)]^2 + [g'(t)]^2}\ dt, \quad g(t) \geq 0$$

Revolution about the y-axis:

$$s = 2\pi\int_a^b f(t)\sqrt{[f'(t)]^2 + [g'(t)]^2}\ dt, \quad f(t) \geq 0$$

Find **relative extrema** of a polar graph by setting $\frac{dr}{dq}$ equal to zero. 12.4 p701

Polar Coordinates are (r, θ) where r is the distance from the origin (also called the **pole**) and θ is the counterclockwise rotation in radians. 12.3 p694 The initial ray is called the **polar axis**.

Coordinate Conversion: 12.3 p696 The polar coordinates (r, θ) are related to the rectangular coordinates (x, y) as follows:

$x = r\cos q$	$y = r\sin q$	(convert to polar)
$\tan q = \frac{y}{x}$	$r^2 = x^2 + y^2$	(convert to rectangular)

Symmetry in Polar Coordinates: 12.3 p699 The graph of a polar equation is symmetric with respect to the following if the indicated substitution produces an equivalent equation.

1. The line $\theta = \pi/2$ (vertical axis)	Replace (r, θ) by $(r, \pi - \theta)$ **or** $(-r, -q)$
2. The polar axis (horizontal axis)	Replace (r, θ) by $(r, -\theta)$ **or** $(-r, p - q)$
3. The pole (origin)	Replace (r, θ) by $(r, \pi + \theta)$ **or** $(-r, \theta)$

Polar Slope: 12.4 p702 If f is a differentiable function of θ, then the **slope** of the tangent line to the graph of $r = f(\theta)$ at the point (r, θ) is

$$\frac{dy}{dx} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}, \text{ provided } \frac{dx}{d\theta} \neq 0 \text{ at } (r, \theta)$$

Tangent Lines at the Pole: 12.4 p703 If $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, then the line $\theta = \alpha$ is tangent at the pole to the graph of $r = f(\theta)$.

Area of a Circular Sector: 12.5 p708 $A = \frac{1}{2}\theta r^2$, provided θ is measured in radians.

Area in Polar Coordinates: 12.5 p709 If f is continuous and nonnegative on the interval $[\alpha, \beta]$, then the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is given by

$$A = \frac{1}{2}\int_\alpha^\beta [f(\theta)]^2 d\theta = \frac{1}{2}\int_\alpha^\beta r^2 d\theta$$

Finding points of intersection: 12.5 p711 Points of intersection may be found by solving the equations simultaneously. However, because a point can be represented in different ways in polar coordinates, it may be necessary to graph the equations in order to recognize **all** points of intersection.

Arc Length of a Polar Curve: 12.5 p712 Let f be a function whose derivative is continuous in an interval $\alpha \leq \theta \leq \beta$. The length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$s = \int_a^b \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\ d\theta$$

Area of a **Surface** of Revolution: 12.5 p714 Let f be a function whose derivative is continuous in an interval $\alpha \leq \theta \leq \beta$. The area of the surface formed by revolving the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ about the indicated line is as follows:

About the polar axis:

$$s = 2\pi\int_a^b f(\theta)\sin\theta\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\ d\theta$$

About the line $q = \frac{p}{2}$:

$$s = 2\pi\int_a^b f(\theta)\cos\theta\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\ d\theta$$

Classification of conics by eccentricity: 12.6 p716 The locus of a point in the plane whose distance from a fixed point (*focus*) has a constant ratio to its distance from a fixed line (*directrix*) is a conic. The constant ratio e is the **eccentricity** of the conic.

$$e = \frac{c}{a} = \frac{\text{distance from center to focus}}{\text{distance from center to vertex}}$$

$$e = \frac{\text{distance from vertex to focus}}{\text{distance from vertex to directrix}}$$

1. The conic is an ellipse if $0 < e < 1$.
2. The conic is a parabola if $e = 1$.
3. The conic is a hyperbola if $e > 1$.

Polar Equations for Conics: 12.6 p717 The graph of a polar equation of the form

$$r = \frac{ed}{1 \pm e\cos\theta} \quad \text{or} \quad r = \frac{ed}{1 \pm e\sin\theta}$$

is a conic, where $e > 0$ is the eccentricity and $|d|$ is the distance between the focus at the pole and its corresponding directrix.

$$d = \frac{a-c}{e} + (a-c) \qquad de^2 = c(1-e^2)$$

Horizontal directrix above the pole: $r = \frac{ed}{1+e\sin\theta}$	Horizontal directrix below the pole: $r = \frac{ed}{1-e\sin\theta}$
Vertical directrix to the left of the pole: $r = \frac{ed}{1-e\cos\theta}$	Vertical directrix to the right of the pole: $r = \frac{ed}{1+e\cos\theta}$

Equation of an Elipse: 12.6 p723

Rectangular: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ Polar: $r^2 = \frac{b^2}{1-e^2\cos^2\theta}$

Equation of a Hyperbola:

Rectangular: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ Polar: $r^2 = \frac{-b^2}{1-e^2\cos^2\theta}$

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