INFINITE SERIES
Chapter 10

A **sequence** $\{a_n\}$ is a function whose domain is the set of positive integers n. The **terms** of the sequence are the values a_1, a_2 ,a_3 . . . , a_n. Sequences that have a finite limit are said to **converge** and sequences that do not **diverge**. A sequence is **monotonic** if its terms are all either *nondecreasing* or *nonincreasing*. A sequence is **bounded** if there is a positive number M such that $|a_n| \le M$ for all n. M is the **upper bound** of the sequence. If a sequence is *bounded* and *monotonic*, then it **converges**. Otherwise, who knows? 10.1 p559

Factorials: 0! = 1, 1! = 1, 2! = 1 x 2, 3! = 1 x 2 x 3, etc.

Definition of Convergent and Divergent Series: 10.2 p572 If the sum S_n of the terms of a series converges to some number S, then the series converges. If S_n diverges, then the series diverges.

Summary of Series Tests 10.2 - 10.6

Convergence/Divergence Testing:

- Telescoping Series Test: Must be in the form $(b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \ldots$
- Geometric Series Test: Must be in the form ar^n.
- Integral Test: Must be positive, continuous, and decreasing for $n \ge 1$.
- p-Series Test: Must be in the form $1/n^p$.
- Ratio Test: Must be a series with non-zero terms.
- Root Test: Helps if the series has a power of n.

Comparison Testing:

- Direct Comparison Test: Must know of a larger converging series to prove convergence or a smaller diverging series to prove divergence.
- Limit Comparison Test: Both series must be greater than 0.

Convergence Testing:

- Alternating Series Test: Must be an alternating series whose sequence goes to 0.

Divergence Testing:

- n^{th} Term Test: Sequence must not converge to 0.

A series is a **geometric series** if the ratios between adjacent terms are equal. A geometric series with ratio r : 10.2 p574

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots + ar^n + \ldots, \quad a \neq 0$$

$$r = \frac{a_n}{a_{n-1}} = \frac{\text{term}}{\text{previous term}}$$

Telescoping Series Test: 10.2 p573 A telescoping series is of the form $(b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \ldots + (b_{n-1} - b_n)$. All of the terms cancel except b_1 and b_n. The series converges if b_n is a finite number. If it converges, the sum is $b_1 - b_n$.

Geometric Series Test: 10.2 p574

If $|r| \ge 1$, this series *diverges*.

If $0 < |r| < 1$, this series *converges* to the sum

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad 0 < |r| < 1$$

If $\sum a_n = A$, $\sum b_n = B$, and c is a real number, then the following series converge to the indicated sums.

$$\sum_{n=1}^{\infty} ca_n = cA \qquad \sum_{n=1}^{\infty} (a_n + b_n) = A + B$$

$$\sum_{n=1}^{\infty} (a_n - b_n) = A - B$$

If the series $\sum a_n$ converges, then the sequence $\{a_n\}$ converges to 0. 10.2 p576

n^{th} Term Test: 10.2 p577 If the sequence $\{a_n\}$ does not converge to 0, that is, if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum a_n$ diverges.

Integral Test: 10.3 p581 If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge.

***p*-Series Test**: 10.3 p583

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots$$

where p is a positive constant. The series converges if $p > 1$, and diverges if $0 < p \leq 1$. When $p = 1$, the series is called the **harmonic series**.

Direct Comparison Test: 10.4 p586 Where $0 \leq a_n \leq b_n$ for all $n > N$. In other words, we can ignore the first several terms of the series:

If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Limit Comparison Test: 10.4 p589 $a_n > 0$, $b_n > 0$, and $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = L$ where L is finite and positive. Then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge. If the series is a fraction with a larger power in the denominator, try comparing with a p-series having the same ratio of powers.

Alternating Series Test: 10.5 p593 If $a_n > 0$, then the alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ and $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converge, provided that the following two conditions are met. Note that here a_n is the portion of the term that remains when we remove the part that makes it alternating.

1) $a_{n+1} \leq a_n$, for all $n > N$. In other words, the terms are nonincreasing and the first few terms may be ignored.
2) $\lim_{n\to\infty} a_n = 0$

Remember that if the derivative of a function is always 0 or negative, then the function is nonincreasing.

Alternating Series Remainder: 10.5 p595 If a convergent alternating series satisfies the condition $a_{n+1} \leq a_n$, [in other words it is nonincreasing] then the absolute value of the remainder R_N involved in approximating the sum S by S_N is less than (or equal to) the first neglected term. That is,

$$|S - S_N| = |R_N| \leq a_{N+1}$$

What this means is that the approximate sum is equal to the sum of the first N terms $\pm$ the next term.

Absolute Convergence: 10.5 p596 If the series $\sum |a_n|$ converges, then the series $\sum a_n$ also converges.

Conditional Convergence: 10.5 p597 If the series $\sum a_n$ converges but $\sum |a_n|$ diverges, then $\sum a_n$ is *conditionally convergent*.

Note: To test a convergent alternating series for absolute or conditional convergence, we simply strip the series of its alternating component and test the result for convergence. If it converges, then the alternating series *converges absolutely*; if it diverges, then the alternating series *converges conditionally*.

The Ratio Test: 10.6 p599 Let $\sum a_n$ be a series with nonzero terms.

1) $\sum a_n$ converges if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$.
2) $\sum a_n$ diverges if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$.
3) The Ratio Test is inconclusive if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$.

The Root Test: 10.6 p602

1) $\sum a_n$ converges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$.
2) $\sum a_n$ diverges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$.
3) The Root Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$.

n^{th} Taylor polynomial: 10.7 p608 If f has n derivatives at c, then the polynomial

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the nth Taylor polynomial for f at c.

n^{th} Maclaurin polynomial: 10.7 p608 If f has n derivatives at 0, then the polynomial

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n$$

is called the nth Maclaurin polynomial for f.

Taylor's Theorem: 10.7 p612 If a function f is differentiable through order $n + 1$ in an interval I containing c, then for each x in I, there exists z between s and c such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x)$$

where $R_n(x) = \dfrac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$.

Power Series: 10.8 p617 If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots$$ is called a *power series*. More generally, we call a series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \ldots + a_n(x-c)^n + \ldots$$

a *power series centered at* c, where c is a constant.

Convergence of a Power Series: 10.8 p618 For a power series centered at c, precisely one of the following is true:

1) The series converges only at c.
2) There exists a real number $R > 0$ such that the series converges (absolutely) for $|x - c| < R$, and diverges for $|x - c| > R$.
3) The series converges for all x.

The number R is called the **radius of convergence** of the power series. If the series converges only at c, then we say that the radius of convergence is $R = 0$, and if the series converges for all x, then we say that the radius of convergence is $R = \infty$. The set of all values of x for which the power series converges is called the **interval of convergence** of the power series. Try using the **Ratio Test** or the **Root Test**.

Properties of a function defined by a power series: 10.8 p622 If the function given by

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \ldots$$

has a radius of convergence of $R > 0$, then on the interval $(c - R, c + R)$ f is continuous, differentiable, and integrable. Moreover, the derivative and antiderivative of f are as follows:

$$f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \ldots$$

$$\int f(x)\,dx = C + a_0(x-c) + a_1\frac{(x-c)^2}{2} + a_2\frac{(x-c)^3}{3} + \ldots$$

To Represent a Function with a Power Series: 10.9 p624 Rewrite the function in the form:

$$\frac{a}{1-k(x-c)} \quad \text{where } r = k(x-c)$$

then $a\sum_{n=0}^{\infty} r^n$ is the power series, or is it $\sum_{n=0}^{\infty} ar^n$?

The **steps** for doing this are:

1. Isolate x.
2. Get the minus sign before the x.
3. Get the c term.
4. Get the 1 in the denominator by selecting the value k to use to multiply the fraction by k/k.

To find the *interval of convergence*, solve $|r| < 1$.

Operations with power series: 10.9 p627 Given $f(x) = \sum a_n x^n$ and $g(x) = \sum b_n x^n$, the following properties are true:

$$f(kx) = \sum_{n=0}^{\infty} a_n k^n x^n \qquad f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$$

$$f(x^N) = \sum_{n=0}^{\infty} a_n x^{nN} \qquad f(x)g(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

When adding or subtracting power series, the resulting interval of convergence is the intersection of the two intervals of convergence.

The form of a Convergent power series: 10.10 p631 If f is represented by a power series $f(x) = \sum a_n (x-c)^n$ for all x in an open interval I containing c, then $a_n = f^{(n)}(c)/n!$ and

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \ldots$$

Taylor series for $f(x)$ at c: 10.10 p632

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n = f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\ldots+\frac{f^{(n)}(c)}{n!}(x-c)^n+\ldots$$

Maclaurin series for f: 10.10 p632

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(n)}(0)}{n!}x^n+\ldots$$

Convergence of Taylor series: 10.10 p634 If a function f has derivatives of all orders in an interval I centered at c, then the equality $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n$ holds if and only if $\lim_{n\to\infty} R_n(x)=0$ for every x in I.

Guidelines for finding a Taylor series: 10.10 p636

1) Differentiate $f(x)$ several times and evaluate each derivative at c. $f(c), f'(c), f''(c), f'''(c), \ldots, f^{(n)}(c), \ldots$ Try to recognize a pattern for these numbers.
2) Use the sequence developed in the first step to form the Taylor coefficients $a_n = f^{(n)}(c)/n!$, and determine the interval of convergence for the resulting power series

$$f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\ldots+\frac{f^{(n)}(c)}{n!}(x-c)^n+\ldots$$

3) Within this interval of convergence, determine whether the series converges to $f(x)$.

Power Series for Elementary Functions	Interval of Convergence
$\frac{1}{x} = 1-(x-1)+(x-1)^2-(x-1)^3+(x-1)^4-\ldots+(-1)^n(x-1)^n+\ldots$	$0<x<2$
$\frac{1}{1+x} = 1-x+x^2-x^3+x^4-x^5+\ldots+(-1)^n x^n+\ldots$	$-1<x<1$
$\ln x = (x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\ldots+\frac{(-1)^{n-1}(x-1)^n}{n}+\ldots$	$0<x\le 2$
$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\ldots+\frac{x^n}{n!}+\ldots$	$-\infty<x<\infty$
$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\ldots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}+\ldots$	$-\infty<x<\infty$
$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\ldots+\frac{(-1)^n x^{2n}}{(2n)!}+\ldots$	$-\infty<x<\infty$
$\arctan x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\ldots+\frac{(-1)^n x^{2n+1}}{(2n+1)}+\ldots$	$-1\le x\le 1$
$\arcsin x = x+\frac{x^3}{2\cdot 3}+\frac{1\cdot 3x^5}{2\cdot 4\cdot 5}+\frac{1\cdot 3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7}+\ldots+\frac{(2n)!x^{2n+1}}{(2^n n!)^2(2n+1)}+\ldots$	$-1\le x\le 1$
$(1+x)^k = 1+kx+\frac{k(k-1)x^2}{2!}+\frac{k(k-1)(k-2)x^3}{3!}+\frac{k(k-1)(k-2)(k-3)x^4}{4!}+\ldots$	$-1<x<1$*

*The convergence at $x=\pm 1$ depends on the value k.

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