INTEGRATION TECHNIQUES
Chapter 9

<u>Procedures for working with integrands</u>: 9.1 p497

1) Expand (carry out operations)
2) Separate numerator (around + or - signs)
3) Complete the square (in denominator)
4) Divide (if power in numerator is equal to or greater than power in denominator)
5) Create new terms in numerator by adding and subtracting a number.
6) Use trigonometric identities.
7) Multiply by conjugate/conjugate.

<u>Integration by Parts</u>: 9.2 p.499

$$\int u\,dv = uv - \int v\,du$$

Try letting dv be the most complicated portion of the integrand that fits an integration formula.

9.3 p509

If the power of the **sine** is **odd**, put in the form:

$$\int (\sin^2 x)^k \cos^n x \sin x\,dx$$

and convert to: $\int (1-\cos^2 x)^k \cos^n x \sin x\,dx$

Reserve the $\sin x\,dx$ term, multiply, and let $u = \cos x$.

If the power of the **cosine** is **odd**, put in the form:

$$\int \sin^m x\,(\cos^2 x)^k \cos x\,dx$$

and convert to: $\int \sin^m x\,(1-\sin^2 x)^k \cos x\,dx$

Reserve the $\cos x\,dx$ term, multiply, and let $u = \sin x$.

If the powers of both the **sine** and **cosine** are **even**, use:

$$\sin^2 x = \frac{1-\cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

to first convert the integrand to odd powers of cosine.

9.3 p509

If the power of the **secant** is **even**, put in the form:

$$\int (\sec^2 x)^k \tan^n x \sec^2 x\,dx$$

and convert to: $\int (1+\tan^2 x)^k \tan^n x \sec^2 x\,dx$

Reserve the $\sec^2 x\,dx$ term, multiply, and let $u = \tan x$.

If the power of the **tangent** is **odd**, put in the form:

$$\int \sec^m x\,(\tan^2 x)^k \sec x \tan x\,dx$$

and convert to: $\int \sec^m x\,(\sec^2 x - 1)^k \sec x \tan x\,dx$

Reserve the $\tan x\,dx$ term, multiply, and let $u = \sec x$.

If there are no secant factors and the power of the **tangent** is **even**, put in the form:

$$\int \tan^n x\,(\tan^2 x)dx$$

and convert to: $\int \tan^n x\,(\sec^2 x - 1)dx$

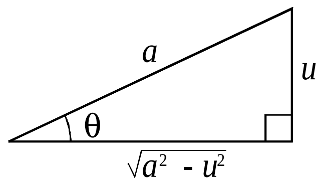
and then: $\int \tan^n x\,(\sec^2 x)dx - \int \tan^n x\,dx$

If the integral is of the form $\int \sec^m x\,dx$. where m is odd, use integration by parts.

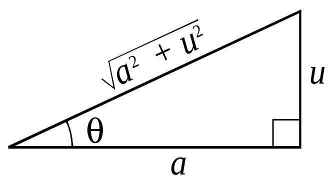
If none of the above applies, try converting to sines and cosines.

Trigonometric Substitution: 9.4 p518 After performing the integration using the substituted values, convert back to terms of x using the right triangles as a guide.

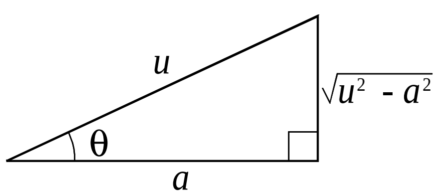
For integrals involving $\sqrt{a^2 - u^2}$, let $u = a\sin\theta$, $du = a\cos\theta\, d\theta$. Then $\sqrt{a^2 - u^2} = a\cos\theta$ where $-\pi/2 \le \theta \le \pi/2$.



For integrals involving $\sqrt{a^2 + u^2}$, let $u = a\tan\theta$, $du = a\sec^2\theta\, d\theta$. Then $\sqrt{a^2 + u^2} = a\sec\theta$ where $-\pi/2 < \theta < \pi/2$.



For integrals involving $\sqrt{u^2 - a^2}$, let $u = a\sec\theta$, $du = a\sec\theta\tan\theta\, d\theta$. Then $\sqrt{u^2 - a^2} = \pm a\tan\theta$ where $0 \le \theta < \pi/2$ or $\pi/2 < \theta \le \pi$. Use the positive value if $u > a$ and the negative value if $u < -a$.



$$\int \sqrt{a^2 - u^2}\, du = \frac{1}{2}\left(a^2 \arcsin\frac{u}{a} + u\sqrt{a^2 - u^2} \right) + C$$

$$\int \sqrt{u^2 - a^2}\, du = \frac{1}{2}\left(u\sqrt{u^2 - a^2} - a^2 \ln\left| u + \sqrt{u^2 - a^2} \right| \right) + C, \quad u > a$$

$$\int \sqrt{u^2 + a^2}\, du = \frac{1}{2}\left(u\sqrt{u^2 + a^2} + a^2 \ln\left| u + \sqrt{u^2 + a^2} \right| \right) + C$$

Partial Fractions: 9.5 p529

The denominator is reduced to simplist terms. Terms with an internal power of 2 require an x-term in the numerator. Terms raised to a power require separate fractions with denominators of each power from 1 to the highest power. To determine the values of the new numerators, multiply each by the value needed to achieve the original common denominator, add them together and set this equal to the original numerator.

$$\int \frac{numerator}{(x+a)(x^2+b)(x+c)^2}\, dx = \int \left(\frac{A}{x+a} + \frac{Bx+C}{x^2+b} + \frac{D}{x+c} + \frac{E}{(x+c)^2} \right) dx$$

$$numerator = A(x^2+b)(x+c)^2 + (Bx+C)(x+a)(x+c)^2 + D(x+a)(x^2+b)(x+c) + E(x+a)(x^2+b)$$

Improper integrals with infinite limits of integration: 9.7p.546

$$\int_a^\infty f(x)\, dx = \lim_{b\to\infty} \int_a^b f(x)\, dx$$

$$\int_{-\infty}^b f(x)\, dx = \lim_{a\to -\infty} \int_a^b f(x)\, dx$$

$$\int_{-\infty}^\infty f(x)\, dx = \int_{-\infty}^c f(x)\, dx + \int_c^\infty f(x)\, dx$$

Improper integrals with an infinite discontinuity: p.549

If f is continuous on the interval $[a, b)$ and has an infinite discontinuity at b, then

$$\int_a^b f(x)\, dx = \lim_{c\to b^-} \int_a^c f(x)\, dx$$

if f is continuous on the interval $(a, b]$ and has an infinite discontinuity at a, then

$$\int_a^b f(x)\, dx = \lim_{c\to a^+} \int_c^b f(x)\, dx$$

If f is continuous on the interval $[a, b]$, except for some c in (a, b) at which f has an infinite discontinuity, then

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

In the first two cases, if the limit exists, then the improper integral **converges**, otherwise, it **diverges**. In the third case, the improper integral on the left **diverges** if either of the improper integrals on the right diverges.