INTEGRATION
Chapter 5

Definition of Integral Notation for Antiderivatives: 5.1 p231

$$\int f(x)\ dx = F(x) + C$$

where: F is an antiderivative of f
C is an arbitrary constant

Integration Formulas: 5.1 p232 To integrate, add one to the exponent, multiply the coefficient by the reciprocal of the new exponent, then add a constant C.

$$\int 0\ dx = C \qquad \int k\ dx = kx + C,\ \ k \neq 0$$

$$\int kf(x)\ dx = k\int F(x)\ dx \qquad \int dx = x + c$$

$$\int [\,f(x) \pm g(x)]\ dx = \int f(x)\ dx \pm \int g(x)\ dx$$

$$\int x^n\ dx = \frac{x^{n+1}}{n+1} + C,\ \ n \neq -1$$

Definition of Sigma Notation: 5.2 p234

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n$$

where: i is the index of summation
a_i is the i^{th} term of the sum
n is the upper bound of summation
1 is the lower bound of summation

Summation Formulas: 5.2 p240

$$\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i\,,\ \ k \text{ is a constant}$$

$$\sum_{i=1}^{n} [a_i \pm b_i] = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Finding upper and lower sums of a region bounded by a function, the x-axis, and two values of x: 5.2 p246

Upper sum: $$\overline{S} = \sum_{i=1}^{n} \frac{(x_2 - x_1)}{n}\, f\left(x_1 + \frac{i(x_2 - x_1)}{n}\right)$$

Lower sum:

$$\underline{s} = \sum_{i=1}^{n} \frac{(x_2 - x_1)}{n}\, f\left(x_1 + \frac{(i-1)(x_2 - x_1)}{n}\right)$$

Where the function $f(x)$ is increasing on the interval (x_1, x_2) and n is the number of divisions between x_1 and x_2. If the function is decreasing, reverse the formulas. $\frac{(x_2 - x_1)}{n}$ represents the width of each division and $f(x_1 + \ldots)$ represents each height involved.

Definition of a Riemann Sum: 5.3 p252 If f is defined on the interval $[a, b]$ and Δ is an arbitrary partition of $[a, b]$, and

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$$

where Δx_1 is the width of the i^{th} subinterval. If c_i is **any** point in the i^{th} subinterval, then the sum

$$\sum_{i-1}^{n} f(c_i)\Delta x_i,\, x_{i-1} \leq c_i \leq x_i$$

is called a **Riemann sum** of f for the partition Δ.

Definition of the Definite Integral: 5.3 p253 If f is defined on the interval $[a, b]$ and the limit of a Riemann sum of f exists, then f is *integrable* on $[a, b]$ and we denote the limit by:

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)dx$$

$\|\Delta\|$ is the width of the largest subinterval or *norm* of the partition. If every subinterval is of equal width then:

$$\|\Delta\| = \Delta x = \frac{b-a}{n} \qquad \text{If not, then: } \frac{b-a}{\|\Delta\|} \leq n$$

$c_i = a + i(\Delta x)$ = the x value where each vertical measurement is taken.

Area of a Region: 5.3 p254 If f is continuous and non-negative on the closed interval [a, b], then the **area** of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$$

Properties of Definite Integrals: 5.3 p256-8

If f is defined at $x = a$, then

$$\int_a^a f(x)dx = 0$$

If f is integrable on [a, b], then

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

Additive Interval Property: If f is integrable on the three closed intervals determined by a, b, and c, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

If f and g are integrable on [a, b] and k is a constant, then:

$$\int_a^b k f(x)dx = k\int_a^b f(x)dx$$

$$\int_a^b \left[f(x) \pm g(x)\right]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

Preservation of Inequality: If f is integrable and nonnegative on the closed interval [a, b], then

$$0 \leq \int_a^b f(x)\,dx$$

And if f and g are integrable on the closed interval [a, b], then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

Involving an **absolute value**: Find the zero of the function and rewrite as a sum.

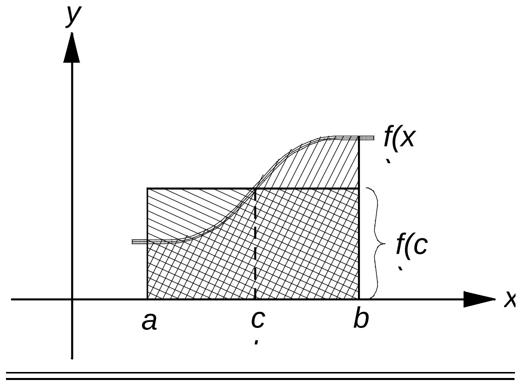
The Fundamental Theorem of Calculus: 5.4 p260
If a function f is continuous on the closed interval [a, b], then:

$$\int_a^b f(x)dx = F(b) - F(a) = \left[F(x)\right]_a^b$$

where F is any function that $F'(x) = f(x)$ for all x in [a, b]. In other words, F is the antiderivative of f. Note that the constant C has been dropped from the antiderivative because it cancels out in subtraction.

The Mean Value Theorem for Integrals: 5.4 p263 If f is continuous on the closed interval [a, b], then there exists a number c in the closed interval [a, b] such that:

$$\int_a^b f(x)dx = f(c)(b-a)$$



Definition of the Average Value of a function on an interval: 5.4 p264 If f is integrable on [a, b], then the **average value** of f on this interval is given by:

$$\frac{1}{b-a}\int_a^b f(x)dx$$

The Second Fundamental Theorem of Calculus: 5.4 p269 If f is continuous on an open interval I containing a, then for every x in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

Antidifferentiation of a Composite Function: 5.5 p269 Let f and g be functions such that $f \circ g$ and g' are continuous on an interval I. If F is an antiderivative of f on I, then

$$\int f(g(x))\,g'(x)\,dx = F(g(x)) + C$$

Change of Variables: 5.5 p272 If we let $u = g(x)$, then $du = g'(x)\,dx$, and the integral above takes the form:

$$\int f(g(x))\,g'(x)\,dx = \int f(u)du = F(u) + C$$

And for *definite integrals*: If the function $u = g(x)$ has a continuous derivative on the closed interval [a, b] and f has an antiderivative over the range of g, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

The above illustrates in equation form how the substitution process below works. The purpose of all this is to allow us to reduce a complex integral to a form that fits a rather limited number of formulas for evaluating integrals.

Guidelines for Integration by Substitution: 5.5 p274

1. Choose a substitution $u = g(x)$. Usually, it is best to choose the *inner* part of a composite function, such as a quantity raised to a power (or root). Sometimes it may be better to let u equal the entire root rather than the value under the root.
2. Take the derivative of u and write in terms of $du = g'(x)\,dx$. In order to get the terms on the right side of this equation to be identical to terms present in the integral, it may be necessary to perform some algebraic manipulation.
3. Rewrite the integral in terms of u and du. It may be necessary here to solve for x in terms of u to complete the substitution.
4. Evaluate the resulting integral.
5. Replace u by $g(x)$ to obtain the antiderivative in terms of x.

The substitution technique needs to be thoroughly understood because it will be used repeatedly in future chapters.

The General Power Rule for Integration: 5.5 p274 If g is a differentiable function of x, then

$$\int [g(x)]^n g'(x)dx = \frac{[g(x)]^{n+1}}{n+1} + C, \quad n \neq -1$$

Equivalently, if $u = g(x)$, then

$$\int u^n du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$$

The Trapezoidal Rule: 5.6 p280 Let f be continuous on $[a, b]$.

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)]$$

where n is the number of subintervals from a to b. This makes the values of x:

$$x_0 = a$$
$$x_1 = a + \tfrac{b-a}{n}$$
$$x_2 = x_1 + \tfrac{b-a}{n}$$
$$\ldots$$
$$x_n = b$$

where $\frac{b-a}{n}$ is the width of the subinterval.

Moreover, as $n \to \infty$, the right-hand side approaches

$$\int_a^b f(x)\,dx$$

Error using the Trapezoidal Rule: 5.6 p284 If f has a continuous second derivative on $[a, b]$, then the error E in approximating $\int_a^b f(x)\,dx$ by the Trapezoidal Rule is:

$$E \leq \frac{(b-a)^3}{12n^2}[\max|f''(x)|], a \leq x \leq b$$

Simpson's Rule (*n* is even): 5.6 p282 Let f be continuous on $[a, b]$.

$$\int_a^b f(x)\,dx \approx \frac{b-a}{3n}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + 4f(x_{n-1}) + f(x_n)]$$

Moreover, as $n \to \infty$, the right-hand side approaches

$$\int_a^b f(x)\,dx\,.$$

Error in Simpson's rule: 5.6 p284 If f has a continuous fourth derivative on $[a, b]$, then the error E in approximating $\int_a^b f(x)\,dx$ by Simpson's Rule is:

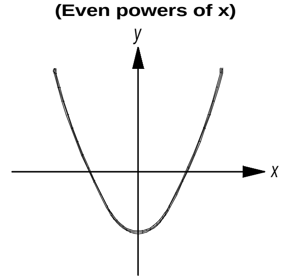
$$E \leq \frac{(b-a)^5}{180n^4}[\max|f^{(4)}(x)|], \quad a \leq x \leq b$$

Theorem for integrals of 2nd Degree Polymonials: 5.6 p282
If $p(x) = Ax^2 + Bx + C$, then

$$\int_a^b p(x)\,dx = \left(\frac{b-a}{6}\right)\left[p(a) + 4p\left(\frac{a+b}{2}\right) + p(b)\right]$$

Even and Odd Functions:

EVEN FUNCTION
(Even powers of x)



ODD FUNCTION
(Odd powers of x)