Graphing Functions
Chapter 4

Relative Extrema

<u>Definition of **Extrema**</u>: I is an interval containing c.

$f(c)$ is a **minimum** if $f(c) \le f(x)$ for each x in I.
$f(c)$ is a **maximum** if $f(c) \ge f(x)$ for each x in I.

<u>Definition of **relative extrema**</u>:
If $f(c)$ is a maximum on (a, b), then $f(c)$ is a **relative maximum** on (a, b).
If $f(c)$ is a minimum on (a, b), then $f(c)$ is a **relative minimum** on (a, b).

<u>Definition of a **critical number**</u>: If f is defined at c, then c is a critical number if $f'(c) = 0$ or if $f'(c)$ does not exist.

<u>Guidelines for finding extrema on $[a, b]$</u>:
1) $f(x)$ must be continuous on $[a, b]$.
2) find critical numbers c on (a, b) and evaluate $f(c)$.
3) find $f(a)$ and $f(b)$.
4) compare $f(a)$, $f(b)$, $f(c)$ to decide maximum and minimum.

<u>Finding intervals where $f(x)$ is **increasing** or **decreasing**</u>:
1) Find any *discontinuities* and *critical numbers*.
2) Set up test intervals using the above.
3) Determine whether $f'(x)$ is positive (*increasing*) or negative (*decreasing*) on each interval. This is called the **First Derivative Test**.

<u>**Rolle's Theorem**</u>: Let f be continuous on a closed interval $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$ then there exists at least one number c in (a, b) such that $f'(c) = 0$ (f has a horizontal tangent line at c, $f(c)$).

<u>**Mean Value Theorem**</u>: If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

In other words, there is some point on the interval where the slope of the tangent line is the same as the slope of a line drawn through the endpoints of the interval.

In terms of *rate of change*, there must be a point on the interval at which the instantaneous rate of change equals the average rate of change on the interval.

A function is **strictly monotonic** on an interval I if it is always increasing or decreasing on I.

Concavity

<u>Definition of **concavity**</u>: Let f be differentiable on an open interval I. The graph of f concaves upward if f' is increasing on I and f concaves downward on I if f' is decreasing on I.

If $f'' > 0$ for x in I, then f concaves upward on I.
If $f'' < 0$ for x in I, then f concaves downward on I.

A **point of inflection** is the point at which concavity changes from upward to downward or visa versa. At this point, f'' is equal to 0 or does not exist.

<u>The **Second Derivative Test**</u>: Let f be a function such that $f'(c) = 0$ and f'' exists on an interval containing c:

If $f''(c) > 0$, then $f(c)$ is a relative minimum.
If $f''(c) < 0$, then $f(c)$ is a relative maximum.
If $f''(c) = 0$, then the test does not apply, and we use the first derivative test.

Asymptotes

Horizontal asymptotes occur when the degree in the numerator is less than or equal to the degree of the denominator. $f(x)$ can have at most two horizontal asymptotes. It is also possible for a function to intersect a horizontal asymptote, but not a vertical asymptote.

If $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$, the $y = L$ is called a **horizontal asymptote** of $f(x)$.

$$y = \frac{Ax^n + \text{(terms of lower degree)}}{Bx^d + \text{(terms of lower degree)}}$$

If $d = n$, $y = A/B$ is a horizontal asymptote.
If $d > n$, $y = 0$ is a horizontal asymptote.
If $d < n$, there is no horizontal asymptote.

Vertical asymptotes occur at the value of x that makes the denominator of the function equal to zero (after the function has been simplified to lowest terms).

Slant asymptotes: If the degree of the numerator is one greater than the denominator, then there exists a slant asymptote. Solve the equation for y and perform the indicated division. Discard the remainder and set y equal to the result. This is the equation of the slant asymptote.

To determine if the curve is above or below the slant asymptote, select an x-value and see whether the *remainder* of the polynomial division is positive (above) or negative (below).

Evaluating Limits

When the result is $\frac{0}{0}$, this is an *indeterminate* form. We suggest rationalizing the denominators.

When the result is $\frac{\infty}{\infty}$, this is an *indeterminate* form. We suggest dividing the numerator and denominator by the highest power of x.

Recheck the original equation to determine the sign of the result.

When the result is ∞ or $-\infty$, then the limit does not exist.

Newton's Method

Newton's Method is used to approximate the zeros of a function. Set up a table as follows:

$$n \qquad x_n \qquad f(x_n) \qquad f'(x_n) \qquad \frac{f(x_n)}{f'(x_n)} \qquad x_n - \frac{f(x_n)}{f'(x_n)}$$

Select a trial value x_n and begin filling in the table. The last value becomes the next x_n. Each successive application of these procedure is called an *iteration*.

One way the method can fail is if the derivative is zero for any x_n in the sequence. This can usually be oversome by choosing a different value for x_1. If successive iterations fail to produce a convergence, then Newton's Method cannot be used. The following condition must be met on the interval containing the zero in order for the method to work:

$$\left| \frac{f(x)f''(x)}{\left[f'(x)\right]^2} \right| < 1$$

Estimation of Error

Implicitly differentiate the equation with respect to the variable with the given error. The differential of that variable with respect to itself will <u>be</u> that given error. For example, if you were to calculate the error in the volume of a sphere with a given error in the radius, then $\frac{dv}{dr}$ would be the result in $\pm$ units^2 and $\frac{dr}{dr}$ would be the error of the radius.

GRAPHING A FUNCTION

1. Find f' and f'' and find the limit of f as $x \to \infty$ and as $x \to -\infty$. Divide both the numerator and denominator by the highest power of x to do this.
2. Determine the symmetry of the function by performing the following tests:

   <u>Symmetry about the origin</u>: Reverse the signs of x and y. If the function is unchanged, symmetry exists.

   <u>Symmetry about the *y* axis</u>: Reverse the sign of x. If the function is unchanged, symmetry exists.

   <u>Symmetry about the *x* axis</u>: Reverse the sign of y. If the function is unchanged, symmetry exists; and, of course, it is not a function.
3. Locate horizontal asymptotes by setting y equal to the result(s) of the limits of f. (Section 4.5) Also refer to notes entitled "Polynomials".
4. Locate vertical asymptotes by finding values for x that make the denominator of the function 0 while the numerator does not equal 0. If values for x result in 0/0, try factoring the fraction or rationalizing the denominator. If the 0/0 result persists then, I believe, a removable discontinuity exists at $x = c$. (Sections 2.3, 2.4)
5. Slant asymptotes occur when the degree of the denominator is one less than the degree of the numerator. Determine the equation of the asymptote by performing the division and disgarding the remainder. Y = the result is the slanted asymptote. (Section 4.5)
6. Determine critical numbers by finding where $f' = 0$ or does not exist. The function must be defined at these numbers. (Section 4.1)
7. Locate the intervals of concavity by determining for what values of x f'' is equal to 0 or is undefined. (Section 4.4)
8. Begin constructing a chart by setting up test intervals by combining values of x for the domain, vertical asymptotes, critical numbers, and intervals of concavity. (Section 4.3, 4.4)
9. Select a value in each interval to determine whether f' is positive or negative in each interval. List continuous intervals over which f' is positive as f increasing and over which f' is negative as f decreasing. (Section 4.1, 4.3)
10. Select a value in each interval to determine whether f'' is positive or negative in each interval. List continuous intervals over which f'' is positive as f concave upward and over which f'' is negative as f concave downward. (Section 4.1, 4.4)
11. Locate any relative maximums by determining points at which the function changes from increasing to decreasing. Locate relative minimums by determining points at which the function changes from decreasing to increasing. (Section 4.1)
12. Locate any points of inflection on the graph by determining points at which the concavity changes. (Section 4.4)
13. Locate any x and y intercepts.