CALCULUS SUMMARY

A quick reference on Exponents, Logarithms, Differentiation, Integration, Power Series

Exponents

$b^x > 0$ $\quad$ $b^x = b^y$ if and only if $x = y$

$b^x b^y = b^{x+y}$ $\quad$ $(b^x)^y = b^{xy}$

$b^x / b^y = b^{x-y}$ $\quad$ $a^x = e^{x \ln a}$

Logarithms

Natural Logarithmic Function $f(x) = \log_e x = \ln x$ The **natural number *e*** ≈ 2.71828182846. To get this number on the calculator, press `1 INV lnx`. $\log_e x$ is written $\ln x$ (read "el - en - ex")

$e = \lim_{x\to 0}(1+x)^{1/x}$ $\quad$ $\ln x = b$ if and only if $e^b = x$

$\lim_{x\to 0^+} \ln x = -\infty$ $\quad$ $\lim_{x\to\infty} \ln x = \infty$.

$\ln e^x = x$ $\quad$ $e^{a \ln b} = b^a$

$\ln xy = \ln x + \ln y$ $\quad$ $\ln \frac{x}{y} = \ln x - \ln y$

$\ln x^y = y \ln x$

Logarithms to other bases:

$y = \log_a x$ if and only if $a^y = x$

$\log_a xy = \log_a x + \log_a y$ $\quad$ $\log_a \frac{x}{y} = \log_a x - \log_a y$

$\log_a x^y = y \log_a x$ $\quad$ $\log_a x = \frac{\log_b x}{\log_b a}$

A **calculator** can be used to evaluate an expression such as $\log_2 14$ by virtue of the fact that it is equivalent to $\ln 14 / \ln 2$.

RULES OF DIFFERENTIATION

(where u is a function of x)

The derivative of a **constant** is 0.

The **power** rule: the derivative of x^n is nx^{n-1}.

The **general power** rule*: $\frac{d}{dx} u^n = nu^{n-1} \cdot u'$

The **constant multiple** rule: $\frac{d}{dx}(c \cdot u) = c \cdot u'$

The **sum and difference** rule: $\frac{d}{dx}(u \pm v) = u' \pm v'$

*The General Power rule is a special case of the Chain rule.

The **product** rule: $\frac{d}{dx}(u \cdot v) = u \cdot v' + u' \cdot v$

The **quotient** rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot u' - u \cdot v'}{v^2}$

The **chain** rule*: $\frac{d}{dx} f(u) = \frac{d}{du} f(u) \cdot \frac{d}{dx} u$

The **absolute value** rule: $\frac{d}{dx}|u| = u' \frac{u}{|u|}, \quad u \neq 0$

Exponential functions: $\frac{d}{dx}(a^x) = (\ln a) a^x$

$\frac{d}{dx}(a^u) = (\ln a) a^u \cdot u'$

The **natural number *e***: $\frac{d}{dx}(e^u) = e^u \cdot u'$

The **natural log**: $\frac{d}{dx}(\ln x) = \frac{1}{x}$

$\frac{d}{dx}(\ln u) = \frac{u'}{u}$ $\quad$ $\frac{d}{dx}(u \cdot \ln x) = \frac{u}{x} + \ln x \cdot u'$

Logarithms to other bases: $\frac{d}{dx}(\log_a x) = \frac{1}{(\ln a) x}$

$\frac{d}{dx}(\log_a u) = \frac{1}{(\ln a) u} \cdot u'$

Trigonometric formulas:

$\frac{d}{dx}(\sin u) = u' \cos u$ $\quad$ $\frac{d}{dx}(\cos u) = -u' \sin u$

$\frac{d}{dx}(\tan u) = u' \sec^2 u$ $\quad$ $\frac{d}{dx}(\sec u) = u' \sec u \tan u$

$\frac{d}{dx}(\cot u) = -u' \csc^2 u$ $\quad$ $\frac{d}{dx}(\csc u) = -u' \csc u \cot u$

$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$ $\quad$ $\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$

$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$ $\quad$ $\frac{d}{dx}[\text{arccot}\, u] = \frac{-u'}{1+u^2}$

$\frac{d}{dx}[\text{arcsec}\, u] = \frac{u'}{|u|\sqrt{u^2-1}}$ $\quad$ $\frac{d}{dx}[\text{arccsc}\, u] = \frac{-u'}{|u|\sqrt{u^2-1}}$

$\frac{d}{dx}(\sinh u) = u' \cosh u$ $\quad$ $\frac{d}{dx}(\cosh u) = u' \sinh u$

Example of the general power rule with trigonometric functions:

$$\frac{d}{dx}(\sin^3 x) = \frac{d}{dx}(\sin x)^3 = 3(\sin x)^2 \cos x = 3\sin^2 x \cos x$$

RULES OF INTEGRATION

The **basic formula**: $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$

Constants: $\int 0 \, dx = C$ $\quad \int dx = x + c$

$\int k \, dx = kx + C \quad \int k f(x) \, dx = k \int f(x) \, dx$

The **sum and difference** rule:

$$\int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx$$

Fractional functions: $\int \frac{1}{x} dx = \ln|x| + C$

$\int \frac{1}{u} du = \ln|u| + C \quad \int \frac{u'}{u} dx = \ln|u| + C$

Exponential functions: $\int a^x dx = \left(\frac{1}{\ln a}\right) a^x + C$

The **natural number *e***: $\int e^x dx = e^x + C$

$\int e^u dx = \frac{1}{u'} \cdot e^u + C \quad \int x e^x dx = (x-1) e^x + C$

$\int x e^{ax} dx = \frac{e^{ax}}{a^2}(ax - 1) + C$

$$\int_0^\infty x^n e^{-ax^2} dx = \begin{cases} \dfrac{\left[(n-1)/2\right]!}{2a^{(n+1)/2}} & \text{for odd } n \\ \dfrac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2^{(n/2)+1} a^{(n/2)}} \sqrt{\dfrac{\pi}{a}} & \text{for even } n \end{cases}$$

Composite function where *u* is a function of *x*:

$$\int f(u) u' \, dx = F(u) + C$$

The **general power rule**: $\int u^n u' dx = \frac{u^{n+1}}{n+1} + C$

Integration by parts; try letting *dv* be the most complicated portion of the integrand that fits an integration formula: $\int u \, dv = uv - \int v \, du$

The **definite integral** where *f*(*x*) is the derivative of *F*(*x*):

$$\int_a^b f(x) dx = F(b) - F(a) = \left[F(x)\right]_a^b$$

Second degree polynomials for $p(x) = Ax^2 + Bx + C$:

$$\int_a^b p(x) \, dx = \left(\frac{b-a}{6}\right)\left[p(a) + 4p\left(\frac{a+b}{2}\right) + p(b)\right]$$

Trigonometric formulas:

$\int \sin u \, dx = -\frac{1}{u'} \cos u + C \quad \int \cos u \, dx = \frac{1}{u'} \sin u + C$

$\int \sec u \, du = \ln|\sec u + \tan u| + C$

$\int \sec^2 u \, du = \tan u + C \quad \int \sec u \tan u \, du = \sec u + C$

$\int \csc u \, du = -\ln|\csc u + \cot u| + C$

$\int \csc^2 u \, du = -\cot u + C$

$\int \csc u \cot u \, du = -\csc u + C$

$\int \tan u \, du = -\ln|\cos u| + C \quad \int \cot u \, du = \ln|\sin u| + C$

$\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C \quad \int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$

$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} arc\sec \frac{|u|}{a} + C$

$\int \sin^2 u \, du = \frac{1}{2} u - \frac{1}{4} \sin 2u + C$

$\int \cos^2 u \, du = \frac{1}{2} u + \frac{1}{4} \sin 2u + C$

$\int \sin^3 u \, du = -\frac{1}{3}(2 + \sin^2 u) \cos u + C$

$\int \cos^3 u \, du = \frac{1}{3}(2 + \cos^2 u) \sin u + C$

Definite Integrals

Natural Number *e*:

$\int_0^\infty e^{-x/\alpha} dx = \alpha \quad \int_0^\infty x e^{-x/\alpha} dx = \alpha^2$

$\int_0^\infty x^2 e^{-x/\alpha} dx = 2\alpha^3 \quad \int_0^\infty x^n e^{-x/\alpha} dx = n! \alpha^{n+1}$

Probability Integrals

For the form $I_n = \int_0^\infty x^n e^{-ax^2} dx$

$I_0 = \frac{\sqrt{\pi}}{2} a^{-1/2} \quad I_1 = \frac{1}{2a} \quad I_2 = \frac{\sqrt{\pi}}{4} a^{-3/2}$

$I_3 = \frac{1}{2} a^{-2} \quad I_4 = \frac{3\sqrt{\pi}}{8} a^{-5/2} \quad I_3 = a^{-3}$

for odd *n*: $I_n = \frac{\left[(n-1)/2\right]!}{2a^{(n+1)/2}}$

for even *n*: $I_n = \frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2^{(n/2)+1} a^{(n/2)}} \sqrt{\frac{\pi}{a}}$

Complex Trigonometric Identities

$\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta}) \quad e^{j\theta} = \cos\theta + j \sin\theta$

$\sin\theta = \frac{1}{j2}(e^{j\theta} - e^{-j\theta})$

Tom Penick tom@tomzap.com www.teicontrols.com/notes 6/11/2001

Quadratic Equation

Given the equation $ax^2 + bx + c = 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Power Series Representation

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$$

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ...$$

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n} = 1 + x^2 + x^4 + x^6 + ..., \quad |x| < 1$$

$$\frac{1}{1+x} \approx 1 - x, \quad |x| \ll 1$$

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x, \quad |x| \ll 1$$