MICROWAVE AND RADIO FREQUENCY ENGINEERING

INDEX

TRANSMISSION LINES

TELEGRAPHER'S EQUATIONS

$$(1)\quad \frac{\partial V}{\partial z} = -L\frac{\partial I}{\partial t} \qquad (2)\quad \frac{\partial I}{\partial z} = -C\frac{\partial V}{\partial t}$$

By taking the partial derivative with respect to *z* of equation 1 and partial with respect to *t* of equation 2, we can get:

$$(\text{i})\quad \frac{\partial^2 V}{\partial z^2} = LC\frac{\partial^2 V}{\partial t^2} \qquad (\text{ii})\quad \frac{\partial^2 I}{\partial z^2} = LC\frac{\partial^2 I}{\partial t^2}$$

SOLVING THE EQUATIONS

To solve the equations (i) and (ii) above, we guess that $F(u) = F(z \pm vt)$ is a solution to the equations. It is found that the unknown constant v is the wave propagation velocity.

$$\boxed{V_{total} = V_+(z - vt) + V_-(v + vt)}$$ where:

z is the position along the transmission line, where the load is at z=0 and the source is at z=-l, with l the length of the line.

v is the **velocity of propagation** $1/\sqrt{LC}$ or ω/β, the speed at which the waveform moves down the line; see p 2

t is time

THE COMPLEX WAVE EQUATION

The general solutions of equations (i) and (ii) above yield the complex wave equations for voltage and current. These are applicable when the excitation is sinusoidal and the circuit is under steady state conditions.

$$\boxed{V(z) = V^+e^{-j\beta z} + V^-e^{+j\beta z}}$$

$$\boxed{I(z) = I^+e^{-j\beta z} + I^-e^{+j\beta z}}$$

$$\boxed{I(z) = \frac{V^+e^{-j\beta z} + V^-e^{+j\beta z}}{Z_0}}$$ where:

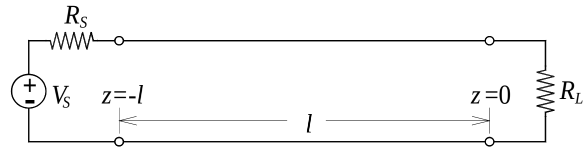
$e^{-j\beta z}$ and $e^{+j\beta z}$ represent wave propagation in the $+z$ and $-z$ directions respectively,

$\beta = \omega\sqrt{LC} = \omega/v$ is the **phase constant,**

$Z_0 = \sqrt{L/C}$ is the **characteristic impedance** of the line. These equations represent the voltage and current **phasors**.

+/- WATCHING SIGNS

By convention z is the variable used to describe position along a transmission line with the origin z=0 set at the load so that all other points along the line are described by **negative** position values.



Ohm's law for right- and left-traveling disturbances:

$$V_+ = I_+Z_0 \qquad V_- = -I_-Z_0$$

v_p VELOCITY OF PROPAGATION [cm/s]

The velocity of propagation is the speed at which a wave moves down a transmission line. The velocity approaches the speed of light but may not exceed the speed of light since this is the maximum speed at which information can be transmitted. But v_p may exceed the speed of light mathematically in some calculations.

$$\boxed{v_p = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\varepsilon\mu}} = \frac{\omega}{\beta}}$$ where:

L = inductance per unit length [H/cm]
C = capacitance per unit length [F/cm]
ε = permittivity of the material [F/cm]
μ = permeability of the material [H/cm]
ω = frequency [radians/second]
β = phase constant

Phase Velocity The velocity of propagation of a TEM wave may also be referred to as the *phase velocity*. The phase velocity of a TEM wave in conducting material may be described by:

$$\boxed{v_p = \omega\delta = \frac{\omega}{k} = c\frac{2\pi\delta}{\lambda_0} = c\frac{1}{\sqrt{\varepsilon_{r\ \text{eff}}}}}$$ where:

δ = skin depth [m]
c = speed of light 2.998×10^8 m/s
λ_0 = wavelength in the material [m]

Z_0 CHARACTERISTIC IMPEDANCE [Ω]

The **characteristic impedance** is the resistance initially *seen* when a signal is applied to the line. It is a physical characteristic resulting from the materials and geometry of the line.

Lossless line: $$\boxed{Z_0 \equiv \sqrt{\frac{L}{C}}} = \frac{V_+}{I_+} = -\frac{V_-}{I_-}$$

Lossy line: $$\boxed{Z_0 \equiv \sqrt{\frac{R + j\omega L}{G + j\omega C}} = |Z_0| e^{j\phi_z}}$$

L = inductance per unit length [H/cm]
C = capacitance per unit length [F/cm]
V_+ = the forward-traveling (left to right) voltage [V]
I_+ = the forward-traveling (left to right) current [I]
V_- = the reverse-traveling (right to left) voltage [V]
I_- = the reverse-traveling (right to left) current [I]
R = the line resistance per unit length [Ω/cm]
G = the conductance per unit length [Ω^{-1}/cm]
ϕ = phase angle of the complex impedance [radians]

y_0 CHARACTERISTIC ADMITTANCE [Ω^{-1}]

The **characteristic admittance** is the reciprocal of the characteristic impedance.

$$\boxed{y_0 \equiv \sqrt{\frac{C}{L}}} = \frac{I_+}{V_+} = -\frac{I_-}{V_-}$$

ρ REFLECTION COEFFICIENT

The **reflection coefficient** is the ratio of reflected voltage to the forward-traveling voltage, a value ranging from –1 to +1 which, when multiplied by the wave voltage, determines the amount of voltage reflected at one end of the transmission line.

$$\boxed{\rho \equiv \frac{V_-}{V_+} = -\frac{I_-}{I_+}}$$

A reflection coefficient is present at each end of the transmission line:

$$\boxed{\rho_{\text{source}} = \frac{R_S - z_0}{R_S + z_0}} \qquad \boxed{\rho_{\text{load}} = \frac{R_L - z_0}{R_L + z_0}}$$

τ TRANSMISSION COEFFICIENT

The **transmission coefficient** is the ratio of total voltage to the forward-traveling voltage, a value ranging from 0 to 2.

$$\boxed{\tau \equiv \frac{V_{total}}{V_+}} = 1 + \rho$$

TOF TIME OF FLIGHT [s]

The **time of flight** is how long it takes a signal to travel the length of the transmission line

$$\boxed{TOF \equiv \frac{l}{v}} = l\sqrt{LC} = \sqrt{L_{TOT} C_{TOT}}$$

l = length of the transmission line [cm]
v = the velocity of propagation $1/\sqrt{LC}$, the speed at which the waveform moves down the line
L = inductance per unit length [H/cm]
C = capacitance per unit length [F/cm]
L_{TOT} = total inductance [H]
C_{TOT} = total capacitance [F]

DERIVED EQUATIONS

$$V_+ = z_0 I_+ = \left(V_{TOT} + I_{TOT} z_0\right)/2$$
$$V_- = -z_0 I_- = \left(V_{TOT} - I_{TOT} z_0\right)/2$$
$$I_+ = y_0 V_+ = \left(I_{TOT} + V_{TOT} y_0\right)/2$$
$$I_- = -y_0 V_- = \left(I_{TOT} - V_{TOT} y_0\right)/2$$

C_n FOURIER SERIES

The function $x(t)$ must be periodic in order to employ the Fourier series. The following is the exponential Fourier series, which involves simpler calculations than other forms but is not as easy to visualize as the trigonometric forms.

$$\boxed{C_n = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)\, e^{-jn\omega_0 t}\, dt}$$

C_n = amplitude
T = period [s]
t = time [s]
n = the harmonic (an integer)
ω_0 = frequency $2\pi/T$ [radians]

The function $x(t)$ may be delayed in time. All this does in a Fourier series is to shift the phase. If you know the C_ns for $x(t)$, then the C_ns for $x(t-\alpha)$ are just $C_n e^{-jn\omega 0\alpha}$. (Here, C_ns is just the plural of C_n.)

C CAPACITANCE [F]

$$v(t)=\frac{1}{C}\int_0^t i\,d\tau+v(0) \qquad I_{cap}=C\frac{dV_{cap}}{dt}$$

$$\boxed{v(t)=v_f+(v_0-v_f)e^{-t/\tau}} \qquad i(t)=i_f+(i_0-i_f)e^{-t/\tau}$$

$$P(t)=i_0{}^2Re^{-2t/\tau}$$

$v(t)$ = voltage across the capacitor, at time t [V]

v_f = final voltage across the capacitor, steady-state voltage [V]

v_0 = initial voltage across the capacitor [V]

t = time [s]

τ = the time constant, RC [seconds]

C = capacitance [F]

Natural log: $\ln x=b \Leftrightarrow e^b=x$

C PARALLEL PLATE CAPACITANCE

$$C=\frac{\varepsilon A}{h} \qquad C_{\text{per unit length}}=\frac{\varepsilon A}{lh}=\frac{\varepsilon wl}{lh}=\frac{\varepsilon w}{h}$$

ε = permittivity of the material [F/cm]

A = area of one of the capacitor plates [cm^2]

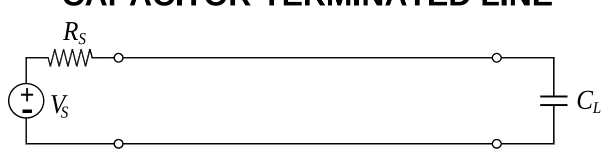
h = plate separation [cm]

w = plate width [cm]

l = plate length [cm]

C = capacitance [F]

CAPACITOR-TERMINATED LINE



Where the incident voltage is $V_+=V_0\left(1-e^{-t/\tau_0}\right)$,

$$V_{cap}=V_++V_-=V_0\left(2+\frac{2\tau_1}{\tau_0-\tau_1}e^{-t/\tau_1}-\frac{2\tau_0}{\tau_0-\tau_1}e^{-t/\tau_0}\right)$$

V_0 = final voltage across the capacitor [V]

t = time [s]

τ_0 = time constant of the incident wave, RC [s]

τ_1 = time constant effect due to the load, Z_0C_L [s]

C = capacitance [F]

SMITH CHART

First **normalize the load impedance** by dividing by the characteristic impedance, and find this point on the chart.

When working in terms of **reactance** X, an inductive load will be located on the top half of the chart, a capacitive load on the bottom half. It's the other way around when working in terms of **susceptance** B [Siemens].

Draw a straight line from the center of the chart through the normalized load impedance point to the edge of the chart.

Anchor a compass at the center of the chart and draw a circle through the normalized load impedance point. Points along this circle represent the normalized impedance at various points along the transmission line. Clockwise movement along the circle represents movement from the load toward the source with one full revolution representing 1/2 wavelength as marked on the outer circle. The two points where the circle intersects the horizontal axis are the voltage maxima (right) and the voltage minima (left).

The point opposite the impedance (180° around the circle) is the **admittance** Y [Siemens]. The reason admittance (or susceptibility) is useful is because admittances in parallel are simply added. (**Admittance** is the reciprocal of impedance; **susceptance** is the reciprocal of reactance.)

$$\boxed{\Gamma(z)=\Gamma_L e^{j2\beta z}}$$

$$e^{j2\beta z}=1\angle 2\beta z$$

$$\boldsymbol{\Gamma}(z)=\frac{\mathbf{Z}(z)-1}{\mathbf{Z}(z)+1}$$

$$\mathbf{Z}_L=\frac{\Gamma_L-1}{\Gamma_L+1} \qquad \boxed{\mathbf{Z}=\frac{Z_L}{Z_0}}$$

z = distance from load [m]

$j=\sqrt{-1}$

ρ = magnitude of the reflection coefficient

β = phase constant

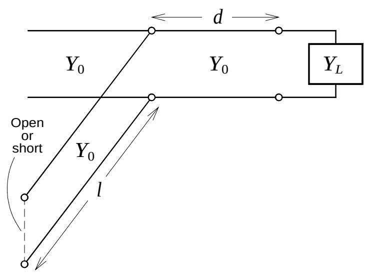
Γ = reflection coefficient

Z = normalized impedance [Ω]

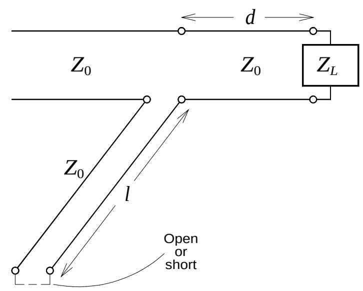
SINGLE-STUB TUNING

The basic idea is to connect a line stub in parallel (shunt) or series a distance d from the load so that the imaginary part of the load impedance will be canceled.

Shunt-stub: Select d so that the admittance Y looking toward the load from a distance d is of the form $Y_0 + jB$. Then the stub susceptance is chosen as $-jB$, resulting in a matched condition.

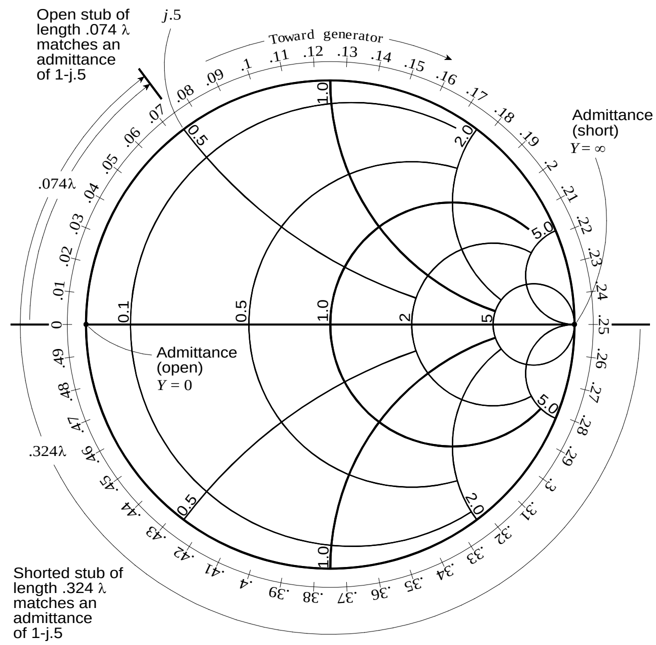


Series-stub: Select d so that the admittance Z looking toward the load from a distance d is of the form $Z_0 + jX$. Then the stub susceptance is chosen as $-jX$, resulting in a matched condition.



FINDING A STUB LENGTH

Example: Find the lengths of open and shorted shunt stubs to match an admittance of 1-j0.5. The admittance of an open shunt (zero length) is $Y=0$; this point is located at the left end of the Smith Chart x-axis. We proceed clockwise around the Smith chart, i.e. away from the end of the stub, to the +j0.5 arc (the value needed to match –j0.5). The difference in the starting point and the end point on the wavelength scale is the length of the stub in wavelengths. The length of a shorted-type stub is found in the same manner but with the starting point at $Y=\infty$.



In this example, all values were in units of admittance. If we were interested in finding a stub length for a series stub problem, the units would be in impedance. The problem would be worked in exactly the same way. Of course in impedance, an open shunt (zero length) would have the value $Z=\infty$, representing a point at the right end of the x-axis.

LINE IMPEDANCE [Ω]

The impedance seen at the source end of a lossless transmission line:

$$Z_{in} = Z_0 \frac{1+\rho}{1-\rho} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)}$$

For a lossy transmission line:

$$Z_{in} = Z_0 \frac{Z_L + Z_0 \tanh(\gamma l)}{Z_0 + Z_L \tanh(\gamma l)}$$

Line impedance is periodic with spatial period λ/2.

$Z_0 = \sqrt{L/C}$, the characteristic impedance of the line. [Ω]
ρ = the reflection coefficient
Z_L = the load impedance [Ω]
$\beta = 2\pi/\lambda$, phase constant
$\gamma = \alpha + j\beta$, complex propagation constant

λ WAVELENGTH [cm]

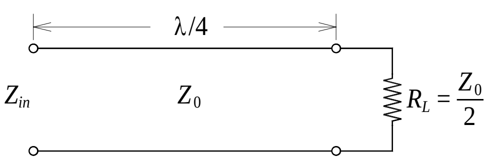
The physical distance that a traveling wave moves during one period of its periodic cycle.

$$\boxed{\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{k} = \frac{v_p}{f}}$$

$\beta = \omega\sqrt{LC} = 2\pi/\lambda$, phase constant
$k = \omega\sqrt{\mu\varepsilon} = 2\pi/\lambda$, wave number
v_p = velocity of propagation [m/s] see p 2.
f = frequency [Hz]

λ/4 QUARTER-WAVE SECTION

A quarter-wave section of transmission line has the effect of inverting the normalized impedance of the load.



To find Z_{in}, we can normalize the load (by dividing by the characteristic impedance), invert the result, and "unnormalize" this value by multiplying by the characteristic impedance.

In this case, the normalized load is $\frac{Z_0}{2} \div Z_0 = \frac{1}{2}$

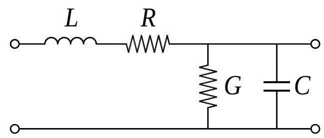
so the normalized input impedance is $\left(\frac{1}{2}\right)^{-1} = 2$

and the actual input impedance is $Z_{in} = 2Z_0$

γ COMPLEX PROPAGATION CONSTANT

The propagation constant for lossy lines, taking into account the resistance and inductance along the line as well as the resistive and capacitive path between the conductors.

$$\gamma = \alpha + j\beta = \sqrt{ZY} = \sqrt{(R + j\omega L)(G + j\omega C)}$$



α = **attenuation constant**, the real part of the complex propagation constant, describes the loss
$\beta = 2\pi/\lambda$, **phase constant**, the complex part of the complex propagation constant
Z = **series impedance** (complex, inductive) per unit length [Ω/cm]
Y = **shunt admittance** (complex, capacitive) per unit length [Ω^{-1}/cm]
R = the resistance per unit length along the transmission line [Ω/cm]
G = the conductance between conductors per unit length [Ω^{-1}/cm]
L = inductance per unit length [H/cm]
C = capacitance per unit length [F/cm]

MODULATED WAVE

Suppose we have a disturbance composed of two frequencies:

$$\sin\left[\left(\omega_0 - \delta\omega\right)t - \left(\beta_0 - \delta\beta\right)z\right]$$

and $$\sin\left[\left(\omega_0 + \delta\omega\right)t - \left(\beta_0 + \delta\beta\right)z\right]$$

where ω_0 is the average frequency and β_0 is the average phase.

Using the identity $2\cos\left(\frac{A-B}{2}\right)\sin\left(\frac{A+B}{2}\right) = \sin A + \sin B$

The combination (sum) of these two waves is

$$\boxed{2\underbrace{\cos\left(\delta\omega t - \delta\beta z\right)}_{\text{envelope}}\underbrace{\sin\left(\omega_0 t - \beta_0 z\right)}_{\text{carrier}}}$$

The envelope moves at the group velocity, see p 7.

δ = "the difference in"…

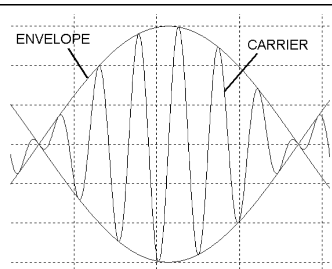
ω_0 = carrier frequency [radians/second]

ω = modulating frequency [radians/second]

β_0 = carrier frequency phase constant

β = phase constant

So the sum of two waves will be a modulated wave having a **carrier** frequency equal to the average frequency of the two waves, and an **envelope** with a frequency equal to half the difference between the two original wave frequencies.



v_g GROUP VELOCITY [cm/s]

The velocity at which the envelope of a modulated wave moves.

$$\boxed{v_g = \frac{\delta\omega}{\delta\beta} = \frac{1}{\sqrt{LC_P}}\sqrt{1 - \frac{\omega_c^2}{\omega^2}}}$$ where:

L = inductance per unit length [H/cm]

C_P = capacitance per unit length [F/cm]

ε = permittivity of the material [F/cm]

μ = permeability of the material, dielectric constant [H/cm]

ω_c = carrier frequency [radians/second]

ω = modulating frequency [radians/second]

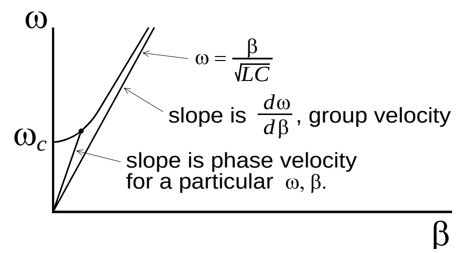
β = phase constant

Also, since β may be given as a function of ω, remember

$$v_g = \left(\frac{d\beta}{d\omega}\right)^{-1}$$

OMEGA - BETA GRAPH

This representation is commonly used for modulated waves.



δ SKIN DEPTH [cm]

The depth into a material at which a wave is attenuated by 1/e (about 36.8%) of its original intensity. This isn't the same δ that appears in the *loss tangent*, tan δ.

$$\boxed{\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}}}$$ where:

α = **attenuation constant**, the real part of the complex propagation constant, describes loss

μ = permeability of the material, dielectric constant [H/cm]

ω = frequency [radians/second]

σ = (*sigma*) conductivity [Siemens/meter] see p12.

Skin Depths of Selected Materials

	60 Hz	1 MHz	1 GHz
silver	8.27 mm	0.064 mm	0.0020 mm
copper	8.53 mm	0.066 mm	0.0021 mm
gold	10.14 mm	0.079 mm	0.0025 mm
aluminum	10.92 mm	0.084 mm	0.0027 mm
iron	0.65 mm	0.005 mm	0.00016 mm

MAXWELL'S EQUATIONS

Maxwell's equations govern the principles of guiding and propagation of electromagnetic energy and provide the foundations of all electromagnetic phenomena and their applications. The time-harmonic expressions can be used only when the wave is sinusoidal.

	STANDARD FORM (Time Domain)	**TIME-HARMONIC** (Frequency Domain)
Faraday's Law	$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\nabla \times \vec{E} = -j\omega \vec{B}$
Ampere's Law*	$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$	$\nabla \times \vec{H} = j\omega \vec{D} + \vec{J}$
Gauss' Law	$\nabla \cdot \vec{D} = \rho_v$	$\nabla \cdot \vec{D} = \rho_v$
no name law	$\nabla \cdot \vec{B} = 0$	$\nabla \cdot \vec{B} = 0$

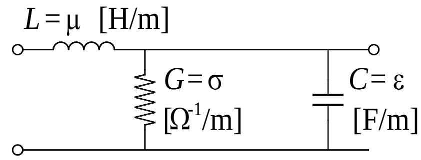
E = electric field [V/m]
B = magnetic flux density [W/m^2 or T] $B = \mu_0 H$
t = time [s]
D = electric flux density [C/m^2] $D = \varepsilon_0 E$
ρ = volume charge density [C/m^3]
H = magnetic field intensity [A/m]
J = current density [A/m^2]

*Maxwell added the $\frac{\partial \vec{D}}{\partial t}$ term to Ampere's Law.

ELECTROMAGNETIC WAVES

MODELING MAXWELL'S EQUATIONS

This is a model of a wave, analogous to a transmission line model.



L = inductance per unit length [H/cm]
μ = permeability of the material, dielectric constant [H/cm]
G = the conductance per unit length [Ω^{-1}/cm]
σ = (*sigma*) conductivity [Siemens/meter]
C = capacitance per unit length [F/cm]
ε = permittivity of the material [F/cm]

propagation constant: $\gamma = \sqrt{(j\omega\mu)(j\omega\varepsilon + \sigma)}$

LOW FREQUENCY

At low frequencies, more materials behave as **conductors**. A wave is considered low frequency when

$\boxed{\omega \ll \frac{\sigma}{\varepsilon}}$ $\frac{\sigma}{\varepsilon}$ is the dielectric relaxation frequency

$\boxed{\eta = \frac{1}{\sigma\delta}(1+j)}$ intrinsic wave impedance, see p 12.

What happens to the complex propagation constant at low frequency? From the wave model above, gamma is

$$\gamma = \sqrt{(j\omega\mu)(j\omega\varepsilon + \sigma)} = \sqrt{j\omega\mu\sigma}\sqrt{1 + \frac{j\omega\varepsilon}{\sigma}}$$

Since both ω and ε/σ are small

$$\gamma = \sqrt{j\omega\mu\sigma}\left(1 + \frac{1}{2}j\omega\frac{\varepsilon}{\sigma}\right) = \sqrt{j\omega\mu\sigma}\,(1)$$

Since $\sqrt{j} = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$

$$\gamma = \sqrt{\omega\mu\sigma}\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{\omega\mu\sigma}{2}} + j\sqrt{\frac{\omega\mu\sigma}{2}}$$

So that, with $\gamma = \alpha + j\beta$

we get $\boxed{\alpha = \sqrt{\frac{\omega\mu\sigma}{2}}}$, $\boxed{\beta = \sqrt{\frac{\omega\mu\sigma}{2}}}$ or $\boxed{\gamma = \frac{1}{\delta}(1+j)}$

α = **attenuation constant**, the real part of the complex propagation constant, describes the loss
β = **phase constant**, the complex part of the complex propagation constant
σ = (*sigma*) conductivity [Siemens/cm]
δ = skin depth [cm]
So the wave is attenuating at the same rate that it is propagating.

HIGH FREQUENCY

At high frequencies, more materials behave as **dielectrics**, i.e. copper is a dielectric in the gamma ray range. A wave is considered high frequency when

$\boxed{\omega \gg \frac{\sigma}{\varepsilon}}$ $\frac{\sigma}{\varepsilon}$ is the dielectric relaxation frequency

$\boxed{\eta = \sqrt{\frac{\mu}{\varepsilon}}}$ intrinsic wave impedance, see p 12.

What happens to the complex propagation constant at high frequency?

$$\gamma = \sqrt{(j\omega\mu)(j\omega\varepsilon + \sigma)} = \sqrt{j\omega\mu\, j\omega\varepsilon\left(1 + \frac{\sigma}{j\omega\varepsilon}\right)}$$

Since both $1/\omega$ and σ/ε are small

$$\gamma = j\omega\sqrt{\mu\varepsilon}\left(1 + \frac{1}{2}\frac{\sigma}{j\omega\varepsilon}\right) \qquad \boxed{\gamma = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} + j\omega\sqrt{\mu\varepsilon}}$$

With $\gamma = \alpha + j\beta$

we get $\boxed{\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}}$, $\boxed{\beta = \omega\sqrt{\mu\varepsilon}}$,

$\tan\delta$ LOSS TANGENT

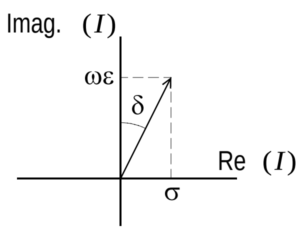
The loss tangent, a value between 0 and 1, is the loss coefficient of a wave after it has traveled one wavelength. This is the way data is usually presented in texts. This is not the same δ that is used for skin depth.

$$\boxed{\tan\delta = \frac{\sigma}{\omega\varepsilon}}$$

Graphical representation of loss tangent:

For a dielectric,
$\tan\delta \ll 1$.

$$\alpha \approx \frac{1}{2}(\tan\delta)\beta = \frac{\pi}{\lambda}\tan\delta$$



$\omega\varepsilon$ is proportional to the amount of current going through the capacitance C.
σ is proportional to the amount current going through the conductance G.

TEM WAVES

Transverse Electromagnetic Waves

Electromagnetic waves that have single, orthogonal vector electric and magnetic field components (e.g., E_x and H_y), both varying with a single coordinate of space (e.g., z), are known as *uniform plane waves* or *transverse electromagnetic (TEM) waves*. TEM calculations may be made using formulas from electrostatics; this is referred to as *quasi-static* solution.

Characteristics of TEM Waves

- The velocity of propagation (always in the *z* direction) is $v_p = 1/\sqrt{\mu\varepsilon}$, which is the speed of light in the material
- There is no electric or magnetic field in the direction of propagation. Since this means there is no voltage drop in the direction of propagation, it suggests that no current flows in that direction.
- The electric field is normal to the magnetic field
- The value of the electric field is η times that of the magnetic field at any instant.
- The direction of propagation is given by the direction of $\mathbf{E} \times \mathbf{H}$.
- The energy stored in the electric field per unit volume at any instant and any point is equal to the energy stored in the magnetic field.

TEM ASSUMPTIONS

Some assumptions are made for TEM waves.

$E_z = 0$ $\qquad H_z = 0$

$\sigma = 0$ $\qquad$ time dependence $e^{j\omega t}$

WAVE ANALOGIES

Plane waves have many characteristics analogous to transmission line problems.

Transmission Lines	Plane Waves
Phase constant $\beta = \omega\sqrt{LC} = \frac{\omega}{v_p} = \frac{2\pi}{\lambda}$	Wave number $k = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{v_p} = \frac{2\pi}{\lambda}$
Complex propagation const. $\gamma = \alpha + j\beta = \sqrt{(R+j\omega L)(G+j\omega C)}$	Complex propagation constant $\gamma = \sqrt{(j\omega\mu)(j\omega\varepsilon+\sigma)}$
Velocity of propagation $v_p = \frac{1}{\sqrt{LC}} = \frac{\omega}{\beta}$	Phase velocity $v_p = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{\omega}{k} = \omega\delta = c\frac{2\pi\delta}{\lambda}$
Characteristic impedance $Z_0 = \sqrt{\frac{L}{C}} = \frac{V_+}{I_+}$	Intrinsic impedance $\eta = \sqrt{\frac{\mu}{\varepsilon}} = \frac{E_{x+}}{H_{y+}}$
Voltage $V(z) = V_+e^{-j\beta z} + V_-e^{j\beta z}$	Electric Field $E_x(z) = E_+e^{-jkz} + E_-e^{jkz}$
Current $I(z) = \frac{1}{Z_0}\left[V_+e^{-j\beta z} - V_-e^{j\beta z}\right]$	Magnetic Field $H_y(z) = \frac{1}{\eta}\left[E_+e^{-jkz} - E_-e^{jkz}\right]$
Line input impedance $Z_{in} = Z_0\frac{Z_L + jZ_0\tan(\beta l)}{Z_0 + jZ_L\tan(\beta l)}$ $Z_{in} = Z_0\frac{Z_L + Z_0\tanh(\gamma l)}{Z_0 + Z_L\tanh(\gamma l)}$	Wave input impedance $\eta_{in} = \eta_0\frac{\eta_L + j\eta_0\tan(kl)}{\eta_0 + j\eta_L\tan(kl)}$ $\eta_{in} = \eta_0\frac{\eta_L + \eta_0\tanh(\gamma l)}{\eta_0 + \eta_L\tanh(\gamma l)}$
Reflection coefficient $\rho = \frac{Z_L - Z_0}{Z_L + Z_0}$	Reflection coefficient $\rho = \frac{\eta_L - \eta_0}{\eta_L + \eta_0}$

k WAVE NUMBER [rad./cm]

The phase constant for the uniform plane wave; the change in phase per unit length. It can be considered a constant for the medium at a particular frequency.

$$\boxed{k = \frac{\omega}{v} = \omega\sqrt{\mu\varepsilon} = \frac{2\pi}{\lambda}}$$

k appears in the phasor forms of the uniform plane wave

$E_x(z) = E_1e^{-jkz} + E_2e^{jkz}$, etc.

k has also been used as in the "*k* of a dielectric" meaning ε_r.

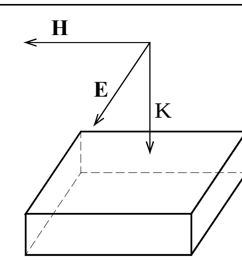
η (eta) INTRINSIC WAVE IMPEDANCE [Ω]

The ratio of electric to magnetic field components. Can be considered a constant of the medium. For free space, $\eta = 376.73\Omega$. The units of η are in ohms.

$$\eta = \frac{E_{x+}}{H_{y+}} = -\frac{E_{y+}}{H_{x+}} \qquad -\eta = \frac{E_{x-}}{H_{y-}} = -\frac{E_{y-}}{H_{x-}}$$

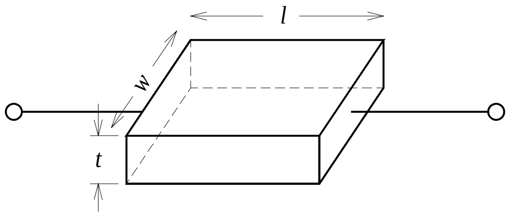
at low frequencies	at high frequencies
$\eta = \frac{1}{\sigma\delta}(1+j)$	$\eta = \sqrt{\frac{\mu}{\varepsilon}}$

When an electromagnetic wave encounters a sheet of conductive material it sees an impedance. K is the direction of the wave, **H** is the magnetic component and **E** is the electrical field. **E** × **H** gives the direction of propagation K.



SHEET RESISTANCE [Ω]

Consider a block of material with conductivity σ.



It's resistance is $R = \dfrac{l}{wt\sigma}\ \Omega.$

If the length is equal to the width, this reduces to

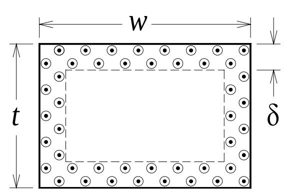
$$R = \frac{1}{t\sigma}\ \Omega.$$

And this is sheet resistance.

HIGH FREQUENCY RESISTANCE [Ω]

When a conductor carries current at high frequency, the electric field penetrates the outer surface only about 1 skin depth so that current travels near the surface of the conductor. Since the entire cross-section is not utilized, this affects the resistance of the conductor.

Cross-section of a conductor showing current flow near the surface:



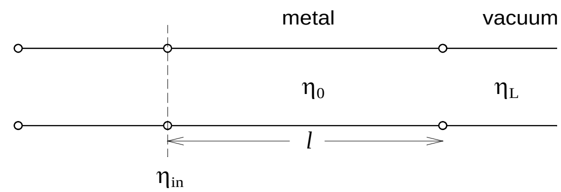
$$\boxed{R \approx \frac{1}{\sigma\delta(\text{perimeter})} = \sqrt{\frac{\omega\mu_0}{2\sigma}}\,\frac{1}{2w+2t}}$$

σ = (*sigma*) conductivity (5.8×10^5 S/cm for copper) [Siemens/meter]
ω = frequency [radians/second]
δ = skin depth [cm]
μ_0 = permeability of free space $\mu_0 = 4\pi\times10^{-9}$ [H/cm]
w = width of the conductor [cm]
t = thickness of the conductor [cm]

η_{in} WAVE INPUT IMPEDANCE [Ω]

The impedance seen by a wave in a medium.

For example, the impedance of a metal sheet in a vacuum:



Note that a transmission line model is used here because it is analogous to a wave traveling in a medium. The "load" is the element most remote in the direction of propagation.

The input impedance is $\eta_{in} = \eta_0 \dfrac{\eta_L + \eta_0 \tanh(\gamma l)}{\eta_0 + \eta_L \tanh(\gamma l)}\ \Omega.$

In this example, *l* is the thickness of a metal sheet. If the metal thickness is much greater than the skin depth, then

$$\tanh(\lambda l) = \tanh\left[\frac{1}{\delta}(1+j)l\right] = \tanh[(\text{big number})(1+j)] \approx 1$$

If *l* is much less than the skin depth δ, then

$$\tanh(\lambda l) = \tanh\left[\frac{1}{\delta}(1+j)l\right] = \tanh[(\text{small number})(1+j)]$$

$$= (\text{same small number})(1+j) = \frac{l}{\delta}(1+j)$$

μ MAGNETIC PERMEABILITY [H/m]

The relative increase or decrease in the resultant magnetic field inside a material compared with the magnetizing field in which the given material is located. The product of the permeability constant and the relative permeability of the material.

$\mu = \mu_0\mu_r$ where $\mu_0 = 4\pi\times10^{-7}$ H/m

Relative Permeabilities of Selected Materials

Air	1.00000037	Mercury	0.999968
Aluminum	1.000021	Nickel	600
Copper	0.9999833	Oxygen	1.000002
Gold	0.99996	Platinum	1.0003
Iron (99.96% pure)	280,000	Silver	0.9999736
Iron (motor grade)	5000	Titanium	1.00018
Lead	0.9999831	Tungsten	1.00008
Manganese	1.001	Water	0.9999912

ε ELECTRIC PERMITTIVITY [F/m]

The property of a dielectric material that determines how much electrostatic energy can be stored per unit of volume when unit voltage is applied, also called the ***dielectric constant***. The product of the constant of permittivity and the relative permittivity of a material.

$$\varepsilon = \varepsilon_0 \varepsilon_r \quad \text{where } \varepsilon_0 = 8.85\times10^{-14} \text{ F/cm}$$

ε_c COMPLEX PERMITTIVITY

$$\varepsilon_c = \varepsilon' - j\varepsilon'' \quad \text{where} \quad \frac{\varepsilon''}{\varepsilon'} = \tan\delta_c$$

In general, both ε' and ε'' depend on frequency in complicated ways. ε' will typically have a constant maximum value at low frequencies, tapering off at higher frequencies with several peaks along the way. ε'' will typically have a peak at the frequency at which ε' begins to decline in magnitude as well as at frequencies where ε' has peaks, and will be zero at low frequencies and between peaks.

ε_r RELATIVE PERMITTIVITY

The permittivity of a material is the relative permittivity multiplied by the permittivity of free space

$$\varepsilon = \varepsilon_r \times \varepsilon_0$$

In old terminology, ε_r is called the "*k* of a dielectric". Glass (SiO_2) at $\varepsilon_r = 4.5$ is considered the division between low *k* and high *k* dielectrics.

Relative Permittivities of Selected Materials

Air (sea level)	1.0006	Polystyrene	2.6
Ammonia	22	Polyethylene	2.25
Bakelite	5	Rubber	2.2-4.1
Glass	4.5-10	Silicon	11.9
Ice	3.2	Soil, dry	2.5-3.5
Mica	5.4-6	Styrofoam	1.03
most metals	~1	Teflon	2.1
Plexiglass	3.4	Vacuum	1
Porcelain	5.7	Water, distilled	81
Paper	2-4	Water, seawater	72-80
Oil	2.1-2.3		

NOTE: Relative permittivity data is given for materials at **low or static frequency conditions**. The permittivity for most materials varies with frequency. The relative permittivities of most materials lie in the range of 1-25. At high frequencies, the permittivity of a material can be quite different (usually less), but will have resonant peaks.

σ CONDUCTIVITY [S/m] or [1/(Ω·m)]

A measure of the ability of a material to conduct electricity, the higher the value the better the material conducts. The reciprocal is *resistivity*. Values for common materials vary over about 24 orders of magnitude. Conductivity may often be determined from skin depth or the loss tangent.

$$\sigma = \frac{n_c q_e^2 \bar{l}}{m_e v_{th}} \text{ S/m} \quad \text{where}$$

n_c = density of conduction electrons (for copper this is 8.45×10^{28}) [m^{-3}]
q_e = electron charge? 1.602×10^{-23} [C]
$\bar{l}$ = $v_{th}t_c$ the product of the thermal speed and the mean free time between collisions of electrons, the average distance an electron travels between collisions [m]
m_e = the effective electron mass? [kg]
v_{th} = thermal speed, usually much larger than the drift velocity v_d. [m/s]

Conductivities of Selected Materials [1/(Ω·m)]

Aluminum	3.82×10^7	Mercury	1.04×10^6
Carbon	7.14×10^4	Nicrome	1.00×10^6
Copper (annealed)	5.80×10^7	Nickel	1.45×10^7
Copper (in class)	6.80×10^7	Seawater	4
Fresh water	$\sim10^{-2}$	Silicon	$\sim4.35\times10^{-4}$
Germanium	~2.13	Silver	6.17×10^7
Glass	$\sim10^{-12}$	Sodium	2.17×10^7
Gold	4.10×10^7	Stainless steel	1.11×10^6
Iron	1.03×10^7	Tin	8.77×10^6
Lead	4.57×10	Titanium	2.09×10^6
		Zinc	1.67×10^7

P POWER [W]

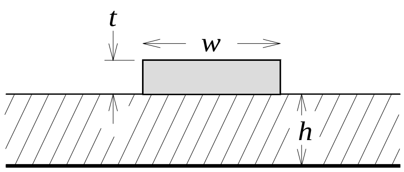
Power is the time rate of change of energy.

Power reflected at a discontinuity: $\% \text{ power} = |\rho|^2 \times 100$

Power transmitted at a discontinuity: $\% \text{ power} = \left(1-|\rho|^2\right)\times 100$

MICROSTRIP CONDUCTORS

How fast does a wave travel in a microstrip? The question is complicated by the fact that the dielectric on one side of the strip may be different from the dielectric on the other side and a wave may travel at different speeds in different dielectrics. The solution is to find an **effective relative permittivity** $\varepsilon_{r\,\mathrm{eff}}$ for the combination.



Some Microstrip Relations

$$Z_0^{\mathrm{air}} = Z_0\sqrt{\varepsilon_{r\,\mathrm{eff}}} \qquad C^{\mathrm{air}} Z_0^{\mathrm{air}} = \sqrt{\varepsilon_0\mu_0}$$

$$L = Z_0^{\mathrm{air}}\sqrt{\varepsilon_0\mu_0} = C^{\mathrm{total}}\left(Z_0\right)^2 \qquad L\,C^{\mathrm{air}} = \varepsilon_0\mu_0$$

$$Z_0 = \sqrt{\frac{L}{C^{\mathrm{total}}}} \qquad Z_0^{\mathrm{air}} = \sqrt{\frac{L}{C^{\mathrm{air}}}}$$

$$\gamma = j\beta = j\omega\sqrt{\varepsilon_0\mu_0}\sqrt{\varepsilon_{r\,\mathrm{eff}}} \qquad \varepsilon_{r\,\mathrm{eff}} = \frac{C^{\mathrm{total}}}{C^{\mathrm{air}}}$$

$$v_p = \frac{1}{\sqrt{\varepsilon_0\mu_0\varepsilon_{r\,\mathrm{eff}}}} = \frac{1}{\sqrt{L\,C^{\mathrm{total}}}}$$

It's difficult to get more than 200Ω for Z_0 in a microstrip.

Microstrip Approximations

$$\varepsilon_{r\,\mathrm{eff}} = \frac{\varepsilon_r + 1}{2} + \frac{\varepsilon_r - 1}{2\sqrt{1 + 12h/w}}$$

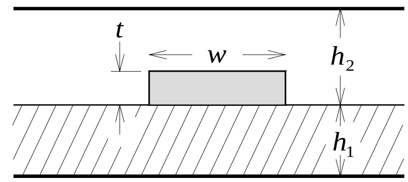
$$Z_0 = \begin{cases} \dfrac{60}{\sqrt{\varepsilon_{r\,\mathrm{eff}}}}\ln\left[\dfrac{8h}{w} + \dfrac{w}{4h}\right], & \text{for } \dfrac{w}{h} \le 1 \\[2ex] \dfrac{120\pi}{\sqrt{\varepsilon_{r\,\mathrm{eff}}}\left[\dfrac{w}{h} + 1.393 + 0.667\ln\left(\dfrac{w}{h} + 1.444\right)\right]}, & \text{for } \dfrac{w}{h} > 1 \end{cases}$$

$$\frac{w}{h} = \begin{cases} \dfrac{8e^{A}}{e^{2A} - 2}, & \dfrac{w}{h} < 2 \\[2ex] \dfrac{2}{\pi}\left\{B - 1 - \ln\left(2B - 1\right) + \dfrac{\varepsilon_r - 1}{2\varepsilon_r}\left[\ln\left(B - 1\right) + 0.39 - \dfrac{0.61}{\varepsilon_r}\right]\right\}, & \dfrac{w}{h} > 2 \end{cases}$$

where $A = \dfrac{Z_0}{60}\sqrt{\dfrac{\varepsilon_r + 1}{2}} + \dfrac{\varepsilon_r - 1}{\varepsilon_r + 1}\left(0.23 + \dfrac{0.11}{\varepsilon_r}\right)$, $B = \dfrac{377\pi}{2Z_0\sqrt{\varepsilon_r}}$

STRIPLINE CONDUCTOR

Also called *shielded microstrip*. The effective relative permittivity is used in calculations.



assuming $w \ge 10h$, $\quad \varepsilon_{r\,\mathrm{eff}} = \dfrac{\varepsilon_{r1}h_1 + \varepsilon_{r2}h_2}{h_1 + h_2}$ where

ε_{r1} = the relative permittivity of the dielectric of thickness h_1.
ε_{r2} = the relative permittivity of the dielectric of thickness h_2.

COPPER CLADDING

The thickness of copper on a circuit board is measured in ounces. 1-ounce cladding means that 1 square foot of the copper weighs 1 ounce. 1-ounce copper is 0.0014" or 35.6 μm thick.

α_d DIELECTRIC LOSS FACTOR [dB/cm]

$$\alpha_d = 8.68\frac{\beta_0\varepsilon_r\left(\varepsilon_{r\,\mathrm{eff}} - 1\right)}{2\sqrt{\varepsilon_{r\,\mathrm{eff}}}\left(\varepsilon_r - 1\right)}\tan\delta$$

α_c CONDUCTOR LOSS FACTOR [dB/cm]

$$\alpha_c = 8.68\frac{R}{2Z_0}, \qquad R = \frac{1}{\sigma\delta(\text{perimeter})} = \sqrt{\frac{\omega\mu_0}{2\sigma}}\frac{1}{(\text{perimeter})}$$

WHEELER'S EQUATION

Another approximation for microstrip calculations is Wheeler's equation.

$$Z_0 = \frac{42.4}{\sqrt{1+\varepsilon_r}} \ln\left(1 + \left\{ \frac{4h}{w'} \left[\frac{14+\frac{8}{\varepsilon_r}}{11} \times \frac{4h}{w'} + \sqrt{\left(\frac{14+\frac{8}{\varepsilon_r}}{11} \times \frac{4h}{w'}\right)^2 + \pi^2 \frac{1+\frac{1}{\varepsilon_r}}{2}} \right] \right\} \right)$$

where $w' = \dfrac{8h\sqrt{\dfrac{7+\frac{4}{\varepsilon_r}}{11}\left[\exp\left(\dfrac{Z_0}{42.4}\sqrt{\varepsilon_r+1}\right)-1\right] + \dfrac{1+\frac{1}{\varepsilon_r}}{0.81}}}{\exp\left(\dfrac{Z_0}{42.4}\sqrt{\varepsilon_r+1}\right)-1}$

NETWORK THEORY

S_{ij} SCATTERING PARAMETER

S_{ij}

observation port excitation port

A scattering parameter, represented by S_{ij}, is a dimensionless value representing the fraction of wave amplitude transmitted from port j into port i, provided that all other ports are terminated with matched loads and only port j is receiving a signal. Under these same conditions, S_{ii} is the reflection coefficient at port i.

To experimentally determine the scattering parameters, attach an impedance-matched generator to one of the ports (***excitation port***), attach impedance-matched loads to the remaining ports, and observe the signal received at each of the ports (***observation ports***). The fractional amounts of signal amplitude received at each port i will make up one column j of the **scattering matrix**. Repeating the process for each column would require n^2 measurements to determine the scattering matrix for an n-port network.

S_{ij} SCATTERING MATRIX

$$\begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1N} \\ S_{21} & S_{22} & \cdots & S_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ S_{N1} & S_{N2} & \cdots & S_{NN} \end{bmatrix}$$

The scattering matrix is an $n \times n$ matrix composed of scattering parameters that describes an n-port network.

The elements of the diagonal of the scattering matrix are reflection coefficients of each port. The elements of the off-diagonal are transmission coefficients, under the conditions outlined in "SCATTERING PARAMETER".

If the network is **internally matched** or **self-matched**, then $S_{11} = S_{22} = \cdots = S_{NN} = 0$, that is, the diagonal is all zeros.

The sum of the squares of each column of a scattering matrix is equal to one, provided the network is lossless.

a_n, b_n INCIDENT/REFLECTED WAVE AMPLITUDES

The parameters a_n and b_n describe the incident and reflected waves respectively at each port n. These parameters are used for power and scattering matrix calculations.

The amplitude of the wave incident to port n is equal to the amplitude of the incident voltage at the port divided by the square root of the port impedance.	$a_n = \dfrac{V_n^+}{\sqrt{Z_{0n}}}$
Amplitude of the wave reflected at port n is equal to the amplitude of the reflected voltage at the port divided by the square root of the port impedance.	$b_n = \dfrac{V_n^-}{\sqrt{Z_{0n}}}$
The scattering parameter is equal to the wave amplitude output at port i divided by the wave amplitude input at port j provided the only source is a matched source at port j and all other ports are connected to matched loads.	$S_{\mathrm{ij}} = \dfrac{b_{\mathrm{i}}}{a_{\mathrm{j}}}$
The relationship between the S-parameters and the a- and b-parameters can be written in matrix form where $\mathbf{S}$ is the scattering matrix and $\mathbf{a}$ and $\mathbf{b}$ are column vectors.	$\mathbf{b} = \mathbf{Sa}$
Power flow into any port is shown as a function of a- and b-parameters.	$P = \dfrac{1}{2}\left(\lvert a\rvert^2 - \lvert b\rvert^2\right)$
The ratio of the input power at port j to the output power at port I can be written as a function of a- and b-parameters or the S-parameter.	$\dfrac{P_{in\,\mathrm{j}}}{P_{out\,\mathrm{i}}} = \dfrac{\lvert a_{\mathrm{j}}\rvert^2}{\lvert b_{\mathrm{j}}\rvert^2} = \dfrac{1}{\lvert S_{\mathrm{ij}}\rvert^2}$

RECIPROCITY

A network is reciprocal when $S_{\mathrm{ij}} = S_{\mathrm{ji}}$ in the scattering matrix, i.e. the matrix is symmetric across the diagonal. Also, $Z_{\mathrm{ij}} = Z_{\mathrm{ji}}$ and $Y_{\mathrm{ij}} = Y_{\mathrm{ji}}$. Networks constructed of “normal materials” exhibit reciprocity.

Reciprocity Theorem:

$$\oint_S \vec{E}_a \times \vec{H}_b \cdot ds = \oint_S \vec{E}_b \times \vec{H}_a \cdot ds$$

E_a and H_b are fields from two different sources.

LOSSLESS NETWORK

A network is lossless when

$$\underline{S}\ \ \underline{S}^{\dagger} = I$$

† means to take the complex conjugate and transpose the matrix. If the network is reciprocal, then the transpose is the same as the original matrix.

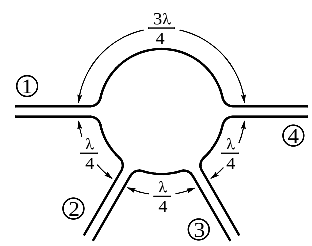
I = a unitary matrix. A **unitary matrix** has the properties:

$$\sum_{k=1}^{N} S_{ki} S_{ki}^{*} = 1 \qquad \sum_{k=1}^{N} S_{ki} S_{kj}^{*} = 0$$

In other words, a column of a unitary matrix multiplied by its complex conjugate equals one, and a column of a unitary matrix multiplied by the complex conjugate of a different column equals zero.

RAT RACE OR HYBRID RING NETWORK

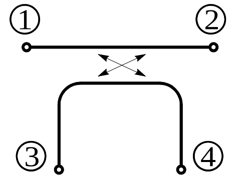
The rat race or hybrid ring network is lossless, reciprocal, and internally matched.



The signal splits upon entering the network and half travels around each side. A signal entering at port 1 and exiting at port 4 travels ¾ of a wavelength along each side, so the signals are in phase and additive. From port 1 to port 3 the signal travels one wavelength along one side and ½ wavelength along the other, arriving a port 3 out of phase and thus canceling. From port 1 to port 2 the paths are ¼ and 5/4 wavelengths respectively, thus they are in phase and additive.

DIRECTIONAL COUPLER

The directional coupler is a 4-port network similar to the rat race. It can be used to measure reflected and transmitted power to an antenna.



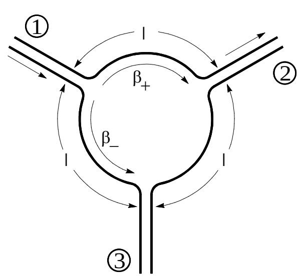
An input at one port is divided between two of the remaining ports. The **coupling factor**, measured in dB, describes the division of signal strength at the two ports. For example if the coupler has a coupling factor of –10 dB, then a signal input at port 1 would appear at port 4 attenuated by 10 dB with the majority of the signal passing to port 2. In other words, 90% of the signal would appear at port 2 and 10% at port 4. (-10 dB means "**10 dB down**" or 0.1 power, -6 dB means 0.25 power, and –3 dB means 0.5 power.) A reflection from port 2 would appear at port 3 attenuated by the same amount. Meters attached to ports 3 and 4 could be used to measure reflected and transmitted power for a system with a transmitter connected to port 1 and an antenna at port 2. The **directivity** of a coupler is a measurement of how well the coupler transfers the signal to the appropriate output without reflection due to the coupler itself; the directivity approaches infinity for a perfect coupler. $\text{directivity} = 10\log\left(p_3 / p_1\right)$, where the source is at port 1 and the load is at port 2.

The directional coupler is **lossless** and **reciprocal**. The scattering matrix looks like this. In a real coupler, the off-diagonal zeros would be near zero due to leakage.

$$\begin{bmatrix} 0 & p & 0 & -q \\ p & 0 & q & 0 \\ 0 & q & 0 & p \\ -q & 0 & p & 0 \end{bmatrix}$$

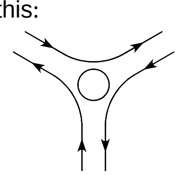
CIRCULATOR

The circulator is a 3-port network that can be used to prevent reflection at the antenna from returning to the source.



Port 3 is terminated internally by a matched load. With a source at 1 and a load at 2, any power reflected at the load is absorbed by the load resistance at port 3. A 3-port network cannot be both lossless and reciprocal, so the circulator is <u>not</u> reciprocal.

Schematically, the circulator may be depicted like this:



The circulator is **lossless** but is <u>not</u> **reciprocal**. The scattering matrix looks like this:

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

MAXWELL'S EQUATIONS, TIME HARMONIC FORM

$$\nabla \times \bar{E} = -j\omega\mu\mathbf{H} \qquad \text{"curl on E"}$$

$$\nabla \times \bar{H} = -j\omega\mu\mathbf{E} \qquad \text{"curl on H"}$$

$$\bar{E} = \left[E_x(x,y)\hat{\mathbf{x}} + E_y(x,y)\hat{\mathbf{y}} + E_z(x,y)\hat{\mathbf{z}}\right]e^{j\omega t - \gamma z}$$

$$\bar{H} = \left[H_x(x,y)\hat{\mathbf{x}} + H_y(x,y)\hat{\mathbf{y}} + H_z(x,y)\hat{\mathbf{z}}\right]e^{j\omega t - \gamma z}$$

From the curl equations we can derive:

(1) $\dfrac{\partial E_z}{\partial y} + \gamma E_y = -j\omega\mu H_x$ (4) $\dfrac{\partial H_z}{\partial y} + \gamma H_y = j\omega\varepsilon E_x$

(2) $-\dfrac{\partial E_z}{\partial x} - \gamma E_x = -j\omega\mu H_y$ (5) $-\dfrac{\partial H_z}{\partial x} - \gamma H_x = j\omega\varepsilon E_y$

(3) $\dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} = -j\omega\mu H_z$ (6) $\dfrac{\partial H_y}{\partial x} - \dfrac{\partial H_x}{\partial y} = j\omega\varepsilon E_z$

From the above equations we can obtain:

(1) & (5) $H_x = \dfrac{1}{\gamma^2 + \omega^2\mu\varepsilon}\left(j\omega\varepsilon\dfrac{\partial E_z}{\partial y} - \gamma\dfrac{\partial H_z}{\partial x}\right)$

(2) & (4) $H_y = \dfrac{1}{\gamma^2 + \omega^2\mu\varepsilon}\left(j\omega\varepsilon\dfrac{\partial E_z}{\partial x} - \gamma\dfrac{\partial H_z}{\partial y}\right)$

(2) & (4) $E_x = -\dfrac{1}{\gamma^2 + \omega^2\mu\varepsilon}\left(-\gamma\dfrac{\partial E_z}{\partial x} + j\omega\mu\dfrac{\partial H_z}{\partial y}\right)$

(1) & (5) $E_y = -\dfrac{1}{\gamma^2 + \omega^2\mu\varepsilon}\left(-\gamma\dfrac{\partial E_z}{\partial y} + j\omega\mu\dfrac{\partial H_z}{\partial x}\right)$

This makes it look like if E_z and H_z are zero, then H_x, H_y, E_x, and E_y are all zero. But since $\infty \times 0 \neq 0$, we could have non-zero result for the TEM wave if $\gamma^2 = -\omega^2\mu\varepsilon \Rightarrow \gamma = j\omega\sqrt{\mu\varepsilon}$. This should look familiar.

WAVE EQUATIONS

From Maxwell's equations and a vector identity on curl, we can get the following wave equations:

$$\nabla^2 \bar{E} = -\omega^2 \mu\varepsilon \bar{E} \qquad \text{"del squared on E"}$$

$$\nabla^2 \bar{H} = \omega^2 \mu\varepsilon \bar{H} \qquad \text{"del squared on H"}$$

The z part or "del squared on E_z" is:

$$\nabla^2 E_z = \frac{\gamma^2 E_z}{\partial x^2} + \frac{\gamma^2 E_z}{\partial y^2} + \frac{\gamma^2 E_z}{\partial z^2} = -\omega^2 \mu\varepsilon E_z$$

Using the separation of variables, we can let:

$$E_z = X(x) \cdot Y(y) \cdot Z(z)$$

We substitute this into the previous equation and divide by $X \cdot Y \cdot Z$ to get:

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{-k_x^2} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{-k_y^2} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{-k_z^2} = \underbrace{-\omega^2 \mu\varepsilon}_{\text{a constant}}$$

Since X, Y, and Z are independent variables, the only way the sum of these 3 expressions can equal a constant is if all 3 expressions are constants.

So we are letting $\frac{1}{Z}\frac{d^2Z}{dz^2} = -k_z^2 \Rightarrow \frac{d^2Z}{dz^2} = -Zk_z^2$

A solution could be $Z = e^{-\gamma z}$

so that $\gamma^2 e^{-\gamma z} = -k_z^2 e^{-\gamma z}$ and $-k_z^2 = \gamma^2$

Solutions for X and Y are found

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k_x^2 \Rightarrow X = A\sin(k_x x) + B\cos(k_x x)$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -k_y^2 \Rightarrow Y = C\sin(k_y y) + D\cos(k_y y)$$

giving us the general solution $k_x^2 + k_y^2 - \gamma^2 = \omega^2 \mu\varepsilon$

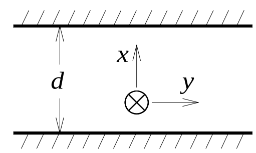
For a particular solution we need to specify initial conditions and boundary conditions. For some reason, initial conditions are not an issue. The unknowns are k_x, k_y, A, B, C, D. The boundary conditions are

$$E_{\tan} = 0 \qquad \frac{\partial H_{\tan}}{\partial n} = 0$$

$E_{\tan}$ = the electric field tangential to a conducting surface
$H_{\tan}$ = the magnetic field tangential to a conducting surface
n = I don't know

TM, TE WAVES IN PARALLEL PLATES

TM, or transverse magnetic, means that magnetic waves are confined to the *transverse plane*. Similarly, TE (transverse electric) means that electrical waves are confined to the transverse plane.



(z direction is into page)

Transverse plane means the plane that is transverse to (perpendicular to) the direction of propagation. The direction of propagation is taken to be in the z direction, so the transverse plane is the x-y plane. So for a TM wave, there is no H_z component (magnetic component in the z direction) but there is an E_z component.

$$\boxed{E_z = A\sin(k_x x)e^{-\gamma z}}$$

A = amplitude [V]

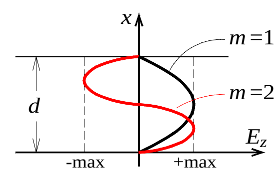
$k_x = \frac{m\pi}{d}$ The magnetic field must be zero at the plate boundaries. This value provides that characteristic. [cm^{-1}]

x = position; perpendicular distance from one plate. [cm]
d = plate separation [cm]
γ = propagation constant
z = position along the direction of propagation [cm]
m = mode number; an integer greater than or equal to 1

$$\boxed{\gamma = \sqrt{-\omega^2 \mu\varepsilon + (kx)^2}}$$

Notice than when $(kx)^2 \geq \omega^2 \mu\varepsilon$, the quantity under the square root sign will be positive and γ will be purely real. In this circumstance, the wave is said to be evanescent. The wavelength goes to infinity; there is no oscillation or propagation. On the other hand, when $(kx)^2 < \omega^2 \mu\varepsilon$, γ is purely imaginary.

The magnitude of E_z is related to its position between the plates and the mode number m. Note that for m = 2 that $d = \lambda$.



GENERAL MATHEMATICAL

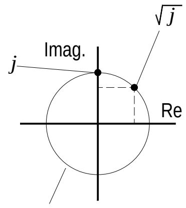
COMPLEX TO POLAR NOTATION

j in polar notation:

$$j = e^{j\frac{\pi}{2}}$$

So we can find the square root of *j*:

$$\sqrt{j} = \sqrt{e^{j\frac{\pi}{2}}} = e^{j\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$



unit circle

dBm DECIBELS RELATIVE TO 1 mW

The decibel expression for power. The logarithmic nature of decibel units translates the multiplication and division associated with gains and losses into addition and subtraction.

0 dBm = 1 mW
20 dBm = 100 mW
-20 dBm = 0.01 mW

$$P(\text{dBm}) = 10\log\left[P(\text{mW})\right]$$

$$P(\text{mW}) = 10^{P(\text{dBm})/10}$$

PHASOR NOTATION

To express a derivative in phasor notation, replace $\frac{\partial}{\partial t}$ with $j\omega$. For example, the Telegrapher's equation $\frac{\partial V}{\partial z} = -L\frac{\partial I}{\partial t}$

becomes $\frac{\partial V}{\partial z} = -Lj\omega I$.

∇ NABLA, DEL OR GRAD OPERATOR

Compare the ∇ operation to taking the time derivative. Where ∂/∂*t* means to take the derivative with respect to time and introduces a s^{-1} component to the units of the result, the ∇ operation means to take the derivative with respect to distance (in 3 dimensions) and introduces a m^{-1} component to the units of the result. ∇ terms may be called space derivatives and an equation which contains the ∇ operator may be called a vector differential equation. In other words ∇**A** is how fast **A** changes as you move through space.

in rectangular coordinates: $\nabla\mathbf{A} = \hat{x}\frac{\partial A}{\partial x} + \hat{y}\frac{\partial A}{\partial y} + \hat{z}\frac{\partial A}{\partial z}$

in cylindrical coordinates: $\nabla\mathbf{A} = \hat{r}\frac{\partial A}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial A}{\partial \phi} + \hat{z}\frac{\partial A}{\partial z}$

in spherical coordinates: $\nabla\mathbf{A} = \hat{r}\frac{\partial A}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial A}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial A}{\partial \phi}$

∇ GRADIENT

$$\boxed{\nabla\vec{\Phi} = -\mathbf{E}}$$

"The gradient of the vector Φ" or "del Φ" is equal to the negative of the electric field vector.

∇Φ is a vector giving the direction and magnitude of the maximum spatial variation of the scalar function Φ at a point in space.

$$\nabla\vec{\Phi} = \hat{\mathbf{x}}\frac{\partial\Phi}{\partial x} + \hat{\mathbf{y}}\frac{\partial\Phi}{\partial y} + \hat{\mathbf{z}}\frac{\partial\Phi}{\partial z}$$

∇· DIVERGENCE

∇· is also a vector operator, combining the "del" or "grad" operator with the dot product operator and is read as "the divergence of". In this form of Gauss' law, where **D** is a density per unit area, with the operators applied, ∇·**D** becomes a density per unit volume.

$$\boxed{\text{div }\mathbf{D} = \nabla\cdot\mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho}$$

D = electric flux density vector **D** = ε**E** [C/m^2]
ρ = source charge density [C/m^3]

∇^2 THE LAPLACIAN

∇^2 is a combination of the divergence and del operations, i.e. $\mathrm{div}(\mathrm{grad}\,\Phi) = \nabla\cdot\nabla\,\Phi = \nabla^2\,\Phi$. It is read as "the LaPlacian of" or "del squared".

$$\boxed{\nabla^2\mathbf{\Phi} = \frac{\partial^2\,\Phi}{\partial\,x^2} + \frac{\partial^2\,\Phi}{\partial\,y^2} + \frac{\partial^2\,\Phi}{\partial\,z^2}}$$

Φ = electric potential [V]

GRAPHING TERMINOLOGY

With *x* being the horizontal axis and *y* the vertical, we have a graph of ***y* versus *x*** or ***y* as a function of *x***. The *x*-axis represents the **independent variable** and the *y*-axis represents the **dependent variable**, so that when a graph is used to illustrate data, the data of regular interval (often this is time) is plotted on the *x*-axis and the corresponding data is dependent on those values and is plotted on the *y*-axis.

HYPERBOLIC FUNCTIONS

$$j\sin\theta = \sinh\left(j\theta\right)$$

$$j\cos\theta = \cosh\left(j\theta\right)$$

$$j\tan\theta = \tanh\left(j\theta\right)$$

TAYLOR SERIES

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x\,, \quad x \ll 1$$

$$\frac{1}{1-x^2} \approx 1 + x^2 + x^4 + x^6 + \cdots\,, \quad |x| < 1$$

$$\frac{1}{1 \pm x} \approx 1 \mp x\,, \quad x \ll 1$$

ELECTROMAGNETIC SPECTRUM

FREQUENCY	WAVELENGTH (free space)	DESIGNATION	APPLICATIONS
< 3 Hz	> 100 Mm		Geophysical prospecting
3-30 Hz	10-100 Mm	ELF	Detection of buried metals
30-300 Hz	1-10 Mm	SLF	Power transmission, submarine communications
0.3-3 kHz	0.1-1 Mm	ULF	Telephone, audio
3-30 kHz	10-100 km	VLF	Navigation, positioning, naval communications
30-300 kHz	1-10 km	LF	Navigation, radio beacons
0.3-3 MHz	0.1-1 km	MF	AM broadcasting
3-30 MHz	10-100 m	HF	Short wave, citizens' band
30-300 MHz 54-72 76-88 88-108 174-216	1-10 m	VHF	TV, FM, police TV channels 2-4 TV channels 5-6 FM radio TV channels 7-13
0.3-3 GHz 470-890 MHz 915 MHz 800-2500 MHz 1-2 2.45 2-4	10-100 cm	UHF "money band"	Radar, TV, GPS, cellular phone TV channels 14-83 Microwave ovens (Europe) PCS cellular phones, analog at 900 MHz, GSM/CDMA at 1900 L-band, GPS system Microwave ovens (U.S.) S-band
3-30 GHz 4-8 8-12 12-18 18-27	1-10 cm	SHF	Radar, satellite communications C-band X-band (Police radar at 11 GHz) K_u-band (dBS Primestar at 14 GHz) K-band (Police radar at 22 GHz)
30-300 GHz 27-40 40-60 60-80 80-100	0.1-1 cm	EHF	Radar, remote sensing K_a-band (Police radar at 35 GHz) U-band V-band W-band
0.3-1 THz	0.3-1 mm	Millimeter	Astromony, meteorology
10^{12}-10^{14} Hz	3-300 μm	Infrared	Heating, night vision, optical communications
3.95×10^{14}-7.7×10^{14} Hz	390-760 nm 625-760 600-625 577-600 492-577 455-492 390-455	Visible light	Vision, astronomy, optical communications Red Orange Yellow Green Blue Violet
10^{15}-10^{18} Hz	0.3-300 nm	Ultraviolet	Sterilization
10^{16}-10^{21} Hz		X-rays	Medical diagnosis
10^{18}-10^{22} Hz		γ-rays	Cancer therapy, astrophysics
> 10^{22} Hz		Cosmic rays	Astrophysics

GLOSSARY

anisotropic materials materials in which the electric polarization vector is not in the same direction as the electric field. The values of ε, μ, and σ are dependent on the field direction. Examples are crystal structures and ionized gases.

complex permittivity ε The imaginary part accounts for heat loss in the medium due to damping of the vibrating dipole moments.

dielectric An insulator. When the presence of an applied field displaces electrons within a molecule away from their average positions, the material is said to be polarized. When we consider the polarizations of insulators, we refer to them as *dielectrics*.

empirical A result based on observation or experience rather than theory, e.g. *empirical data*, *empirical formulas*. Capable of being verified or disproved by observation or experiment, e.g. *empirical laws*.

evanescent wave A wave for which $\beta=0$. α will be negative. That is, γ is purely real. The wave has infinite wavelength—there is no oscillation.

isotropic materials materials in which the electric polarization vector is in the same direction as the electric field. The material responds in the same way for all directions of an electric field vector, i.e. the values of ε, μ, and σ are constant regardless of the field direction.

linear materials materials which respond proportionally to increased field levels. The value of μ is not related to $\mathbf{H}$ and the value of ε is not related to $\mathbf{E}$. Glass is linear, iron is non-linear.

overdamped system in the case of a transmission line, this means that when the source voltage is applied the line voltage rises to the final voltage without exceeding it.

time variable materials materials whose response to an electric field changes over time, e.g. when a sound wave passes through them.

transverse plane perpendicular, e.g. the x-y plane is *transverse* to z.

underdamped system in the case of a transmission line, this means that after the source voltage is applied the line voltage periodically exceeds the final voltage.

wave number k The phase constant for the uniform plane wave. k may be considered a constant of the medium at a particular frequency.