THE LINEAR SYSTEM

Definition, Characteristics, Analysis Methods

DEFINITION: A **linear system** is a system whose output is proportional to its input in the following manner:

1. The total output is the sum of all the component effects. That is, if a cause c_1 has an effect e_1 and if another cause c_2 has an effect e_2, then when both causes act on the system, the effect will be $e_1 + e_2$. This is called **additivity**.

$$\text{If}\quad c_1 \to e_1 \quad\text{and}\quad c_2 \to e_2 \quad\text{then}\quad c_1 + c_2 \to e_1 + e_2\,.$$

2. If a cause increases k-fold then the effect also increases k-fold. This is called **homogeneity**.

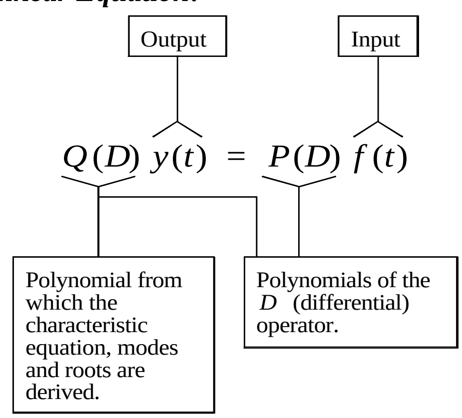
$$\text{If}\quad c \to e \quad\text{then for all } k,\quad kc \to ke\,.$$

The combination of the additivity and homogeneity properties is called **superposition** and can be expressed as:

$$\boxed{k_1c_1 + k_2c_2 \to k_1e_1 + k_2e_2}$$

CHARACTERISTICS OF THE LINEAR EQUATION

***Linear Equation*:**



The closer the input $f(t)$ is to the characteristic mode, the stronger is the system response (output). This is the **resonance phenomenon**.

A system's **time constant** is equal to the width of its impulse response $h(t)$ and represents the time lag before a system fully responds to an input (*rise time*). A large time constant indicates a sluggish system.

Stability is determined by the systems **roots**. If the real part of all roots is negative, the system is said to be **(asymptotically) stable**. That is, in the absence of input the output will tend toward zero. If there are unrepeated roots with the real part equal to zero and no roots having positive real parts then the system is said to be **marginally stable**. That is, it tends toward some non-zero value or bounded oscillatory mode. If there are repeated roots with the real part equal to zero and/or any roots having positive real values then the system is **unstable**. That is, output grows to plus/minus infinity.

TIME-DOMAIN METHOD OF ANALYSIS
Zero-Input + Zero-State

ZERO-INPUT **ZERO-STATE**

$$\text{Total Response} = \sum_{j=1}^{n} c_j e^{\lambda_j t} + f(t) * h(t)$$

system output

result of initial conditions with the input held to zero, composed only of characteristic mode terms

$$\text{Total Response} = \text{zero-input response} + \text{zero-state response}$$

result of system input with initial conditions at zero, composed of both characteristic and non-characteristic mode terms

Zero-Input response

The **zero-input response** is the component of the system output that results from the effect of the initial conditions only (zero input).

Given the linear equation

$$Q(D)\,y(t) = P(D)\,f(t)$$

if the input $f(t)$ equals zero, then the LHS $Q(D)\,y(t)$ must also equal zero. So to calculate the zero-input response $y_0(t)$ we have: $Q(D)y_0(t) = 0$. Since $Q(D)$ must equal 0 for this to be true we now have $Q(D) = 0$. $Q(D)$ is a function involving the *D*-operator (differentiation).

***Characteristic Equation*:** $Q(\lambda) = 0$
where: $Q(\lambda)$ is a polynomial function formed by substituting λ for the *D*-operator in the function $Q(D)$, which is found in the linear equation.

By substituting λ for the *D*-operator in the polynomial function $Q(D)$ and setting the result equal to zero, we obtain the **characteristic equation** $Q(\lambda) = 0$. Solving for λ we get the **characteristic roots** of the linear equation. From the roots, the **characteristic modes** are derived in the form $e^{\lambda t}$. where λ is a root. In the case where there are repeated roots, that is more than one λ with the same value, unique characteristic modes are created in the form $e^{\lambda t}$, $te^{\lambda t}$, $t^2e^{\lambda t}$, etc.

If we have a pair of **complex characteristic roots**, in the form $\alpha \pm j\beta$, the characteristic modes may be written in two ways.

***Complex Roots*:** $\alpha + j\beta$, $\alpha - j\beta$
two possible forms:
1) $c_1 e^{(\alpha + j\beta)t}$, $c_2 e^{(\alpha - j\beta)t}$ with constants c_1 and c_2
2) $ce^{\alpha t}\cos(\beta t + \theta)$ with constants c and θ

The solution is formed by setting $y_0(t)$ equal to the sum of the characteristic modes, each multiplied by a constant. The values of these constants may be determined if initial conditions are known.

Zero-State response

The **zero-state response** is the component of the system output that results from the effect of the input only (zero initial conditions). It is found by convolving the input $f(t)$ with the **unit impulse response** $h(t)$.

(see Convolution.pdf)

***Zero-State Response*:**

$$f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$$

where: $f(t)$ is the system input

$h(t)$ is the **unit impulse response**, i.e. the output of the system in response to a unit impulse at the input

If the system is **causal**, the integration may be taken from 0 to ∞.

The **unit impulse response** $h(t)$ is the output of a system when the input is the unit impulse "function" $\delta(t)$. When the unit impulse response is known, the output can be found for any input function.

(see ImpulseResponse.pdf for a sample problem)

***Unit Impulse Response*:**

$h(t) = b_n\delta(t) + [P(D)y_0(t)]u(t)$, $m \le n$ (most situations)

$h(t) = [P(D)y_0(t)]u(t)$, $n < m$ (not practical)

where, from the system equation $Q(D)y(t) = P(D)f(t)$:

m = the order of the highest order term in the function $P(D)$
n = the order of the highest order term in the function $Q(D)$
b_n = the coefficient of the n^{th} order term in the function $P(D)$. If the term is absent then $b_n = 0$.
$\delta(t)$ = the unit impulse function
$y_0(t)$ = the zero-input response

$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \ge 0 \end{cases}$ (the unit function)

The **unit impulse function** (not a *true* function) represents a single high magnitude pulse occurring at $t = 0$. The width of the function approaches zero and the area under the curve is equal to one. It is defined only at $t = 0$.

***Unit Impulse Function*:**

$\delta(t) = 0, \quad t \ne 0$

$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

THE CLASSICAL METHOD OF ANALYSIS
Natural + Forced Response

$$Q(D)[y_n(t) + y_f(t)] = P(D)f(t),$$

with $Q(D)y_n(t) = 0$

(see Classical.pdf)

system output

sum of the characteristic modes

Total Response = natural response + forced response

composed of only those zero-state response terms which are not characteristic modes

Natural response $y_n(t)$

The natural response consists of all of the characteristic mode terms found in the total response. The natural response has the same characteristic modes as the zero-input response but the arbitrary constants are different.

The natural response is the reaction of the circuit without excitation.

Forced response $y_f(t)$

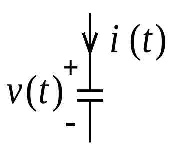
The forced response consists of all non-characteristic mode terms found in the total response.

The forced response is the reaction of the circuit to external excitation.

LINEARITY IN CIRCUIT ELEMENTS
Mapping Inputs to Outputs

THE CAPACITOR

$$i(t) = C\frac{dv(t)}{dt} \qquad (v \text{ to } i)$$



The map that takes v to i is linear. $f(v) = i$, with

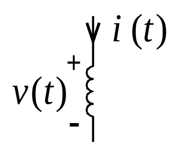
$$f(v)(t) = C\frac{dv(t)}{dt}$$

The map of i to v is **not** linear unless $v(0) = 0$.

$$v(t) = v(0) + \frac{1}{C}\int_0^t i(\tau)d\tau \qquad (i \text{ to } v)$$

and similarly, **THE INDUCTOR**

$$v(t) = L\frac{di(t)}{dt} \qquad (i \text{ to } v)$$



The map that takes i to v is linear. $f(i) = v$, with

$$f(i)(t) = L\frac{di(t)}{dt}$$

The map of v to i is **not** linear unless $i(0) = 0$.

$$i(t) = i(0) + \frac{1}{L}\int_0^t v(\tau)d\tau \qquad (v \text{ to } i)$$