TIME-DOMAIN ANALYSIS OF A CONTINUOUS-TIME SYSTEM

An Example using The Classical Method

THE PROBLEM

Solve the system equation:	$(D^2 + 3D + 2)\,y(t) = Df(t)$
If the input is:	$f(t) = t^2 + 5t + 3$
And the initial conditions are:	$y(0) = 2$ and $Dy(0) = 3$

This problem is also solved using the **frequency-domain method** (Laplace transform); see the document **LaplaceExample.pdf**.

THE SOLUTION

The characteristic polynomial of the system is $\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0$

The roots of the system are $\lambda = -1, -2$

The characteristic modes of the system are $e^{-t},\ e^{-2t}$

The natural response of the system is then: The arbitrary constants a_1 and a_2 will be determined later by the initial conditions.

$$y_n(t) = a_1 e^{-t} + a_2 e^{-2t} \qquad \text{(Eq 4)}$$

The forced response to the input $t^2 + 5t + 3$ is

$$y_f(t) = b_2 t^2 + b_1 t + b_0 \qquad \text{(Eq 5)}$$

$y_f(t)$ satisfies the system equation

$$(D^2 + 3D + 2)\,y_f(t) = Df(t) \qquad \text{(Eq 6)}$$

$Dy_f(t)$ and $D^2y_f(t)$ are found from (Eq 5)

$$Dy_f(t) = \tfrac{d}{dt}(b_2 t^2 + b_1 t + b_0) = 2b_2 t + b_1 \qquad \text{(Eq 7)}$$

$$D^2 y_f(t) = \tfrac{d}{dt}(2b_2 t + b_1) = 2b_2 \qquad \text{(Eq 8)}$$

The derivative of the input $f(t)$ is found

$$Df(t) = \tfrac{d}{dt}(t^2 + 5t + 3) = 2t + 5$$

Taking equation (Eq 6): $(D^2 + 3D + 2)\,y_f(t) = Df(t)$

Rewriting (Eq 6): $D^2 y_f(t) + 3Dy_f(t) + 2y_f(t) = Df(t)$

And substituting (Eq 8), (Eq 7), (Eq 5): $2b_2 + 3(2b_2 t + b_1) + 2(b_2 t^2 + b_1 t + b_0) = 2t + 5$

And regrouping to powers of t: $2b_2 t^2 + (2b_1 + 6b_2)t + (2b_0 + 3b_1 + 2b_2) = 2t + 5$

Tom Penick tomzap@eden.com www.teicontrols.com/notes 05/17/98

Equating coefficients of similar powers on both sides of the expression:

$$\begin{aligned} 2b_2 &= 0 \\ 2b_1 + 6b_2 &= 2 \\ 2b_0 + 3b_1 + 2b_2 &= 5 \end{aligned}$$

Solving these three equations we get: $b_0 = 1,\ b_1 = 1,\ b_2 = 0$

Therefore, from (Eq 5): $y_f(t) = b_2 t^2 + b_1 t + b_0$

We have: $y_f(t) = t + 1,\ \ t > 0$ (Eq 19)

The total system response is the sum of the natural and the forced solutions: $y(t) = y_n(t) + y_f(t)$

So from (Eq 4) and (Eq 19) we have: $y(t) = (a_1 e^{-t} + a_2 e^{-2t}) + (t + 1)$ (Eq 21)

And, taking the derivative: $Dy(t) = -a_1 e^{-t} - 2a_2 e^{-2t} + 1)$ (Eq 22)

So with the initial conditions: $y(0) = 2$ and $Dy(0) = 3$

Substituting into (Eq 21): $2 = a_1 + a_2 + 1$

Substituting into (Eq 22): $3 = -a_1 - 2a_2 + 1$

Solving these two equations we get: $a_1 = 4,\ \ a_2 = -3$

THE ANSWER

Therefore, from (Eq 21): $y(t) = (a_1 e^{-t} + a_2 e^{-2t}) + (t + 1)$

We have: $\boxed{y(t) = 4e^{-t} - 3e^{-2t} + t + 1, \quad t > 0}$