UNDETERMINED COEFFICIENTS

Method for solving a nonhomogeneous second order differential equation

This method is supposed to be simpler than the Variation of Parameters method but is limited to equations where *f*(*x*) is a polynomial, exponential, sine or cosine. For an alternate method see the document VariationOfParameters.pdf.

THE PROBLEM

33. p184 $\boxed{y''+9y = \sin 2x \qquad y(0)=1, \quad y'(0)=0}$

THE APPROACH

To solve this problem we will find a **particular solution** y_p that satisfies the equation and a **complementary solution** y_c of the associated homogeneous equation $y''+9y = 0$. Then we add the results to obtain the general solution $y(x) = y_c + y_p$. Finally we apply the initial conditions to determine the final solution.

THE PARTICULAR SOLUTION

We must search for a value *y* that can satisfy the equation. We take a hint from the term to the right of the equals sign, sin2*x*.

We select a trial term which can still have this form when its derivatives are taken: $y_p = A\sin 2x + B\cos 2x$

Taking the first and second derivatives of this term we have

$$y_p' = 2A\cos 2x - 2B\sin 2x$$

$$y_p'' = -4A\sin 2x - 4B\cos 2x$$

Substituting these terms into the original expression we have: $-4A\sin 2x - 4B\cos 2x + 9A\sin 2x + 9B\cos 2x = \sin 2x$

This reduces to $5A\sin 2x + 5B\cos 2x = \sin 2x$

Matching up the similar terms reveals that *B* is equal to 0 and $A = 1/5$

Therefore our particular solution must actually be $\boxed{y_p = \frac{1}{5}\sin 2x}$

We can verify this by finding its first and second derivatives: $y_p' = \frac{2}{5}\cos 2x \qquad y_p'' = -\frac{4}{5}\sin 2x$

and substituting into the original equation:

$$-\frac{4}{5}\sin 2x + 9\left(\frac{1}{5}\sin 2x\right) = \sin 2x$$

$$\left(\frac{9}{5} - \frac{4}{5}\right)\sin 2x = \sin 2x$$

simplify to show that it satisfies the equation.

THE COMPLEMENTARY SOLUTION

Given our problem expression $y''+9y=\sin 2x$ the characteristic equation is $r^2+9=0$. More about this is available in the document CharacteristicEquations.pdf.

Using the quadratic equation we have $r=\dfrac{0\pm\sqrt{0-36}}{2}$ giving the complex roots $\boxed{r=0\pm 3i}$

This gives the Complementary Solution: $\boxed{y_c=e^{0x}(c_1\cos 3x+c_2\sin 3x)}$

THE GENERAL SOLUTION

Using the formula $y(x)=y_c+y_p$ the general solution is $y(x)=e^{0x}(c_1\cos 3x+c_2\sin 3x)+y_p$

Simplifying and substituting for y_p we get: $\boxed{y(x)=c_1\cos 3x+c_2\sin 3x+\frac{1}{5}\sin 2x}$

APPLYING THE INITIAL CONDITIONS TO DETERMINE THE FINAL SOLUTION

With the general solution $y(x)=c_1\cos 3x+c_2\sin 3x+\frac{1}{5}\sin 2x$ and the initial value $y(0)=1$	With the differential of the general solution $y'(x)=-3c_1\sin 3x+3c_2\cos 3x+\frac{2}{5}\cos 2x$ and the initial value $y'(0)=0$
we have $1=c_1\cos 0+c_2\sin 0+\frac{1}{5}\sin 0$	we have $0=-3c_1\sin 0+3c_2\cos 0+\frac{2}{5}\cos 0$
which simplifies to $1=c_1\cos 0$	which simplifies to $0=3c_2\cos 0+\frac{2}{5}\cos 0$
so that $c_1=1$	so that $c_2=-\frac{2}{15}$

THE FINAL SOLUTION

Substituting these values into the general solution yields the final solution: $\boxed{y(x)=\cos 3x-\frac{2}{15}\sin 3x+\frac{1}{5}\sin 2x}$