FREQUENCY-DOMAIN ANALYSIS OF A CONTINUOUS-TIME SYSTEM

An Example using The Laplace Transform to find the Solution to a Linear System Equation

THE PROBLEM

Solve the system equation:	$(D^2 + 3D + 2)y(t) = Df(t)$
If the input is:	$f(t) = t^2 + 5t + 3$
And the initial conditions are:	$y(0) = 2$ and $Dy(0) = 3$

This problem is also solved using the **classical time-domain method**; see the document **ClassicalExample.pdf**.

THE APPROACH

We wish to solve the system equation for $y(t)$. To solve using the Laplace transform, we replace each element of the system equation with its transform. We use the symbol "$\Leftrightarrow$" to indicate the transformation between the time-domain and frequency-domain (Laplace transform). We then solve for $Y(s)$, which is the Laplace transform of $y(t)$. We convert the answer back to the time domain to get $y(t)$.

THE SOLUTION

Rewrite the system equation:

$$D^2 y(t) + 3Dy(t) + 2y(t) = Df(t)$$

The Laplace transform of the first element is

$$D^2 y(t) \Leftrightarrow s^2 Y(s) - sy(0) - Dy(0)$$
$$= s^2 Y(s) - 2s - 3$$

The Laplace transform of the second element is $3Dy(t) \Leftrightarrow 3sY(s) - 3y(0) \quad = 3sY(s) - 6$

The Laplace transform of the third element is $2y(t) \Leftrightarrow 2Y(s)$

The Laplace transform of the last element is $Df(t) \Leftrightarrow sF(s) - f(0)$

Now we must find $F(s)$ using the Laplace transform

$$F(s) = \int_0^\infty f(t)e^{-st}dt \qquad \text{(Laplace transform)}$$

$$= \int_0^\infty (t^2 + 5t + 3)e^{-st}dt \quad = \int_0^\infty t^2 e^{-st}\,dt + \int_0^\infty 5te^{-st}\,dt + \int_0^\infty 3e^{-st}\,dt$$

$$= \int_0^\infty t^2 e^{-st}\,dt + 5\int_0^\infty te^{-st}\,dt + 3\int_0^\infty e^{-st}\,dt$$

$$= \left[\frac{e^{-st}}{(-s)^3}\left[(-s)^2 t^2 - 2(-s)t + 2\right]\right]_0^\infty + \left[5\frac{e^{-st}}{(-s)^2}\left[(-s)t - 1\right]\right]_0^\infty + 3\frac{-1}{s}\left[e^{-st}\right]_0^\infty$$

$$= \left[\frac{e^{-st}}{-s^3}\left[s^2t^2 + 2st + 2\right]\right]_0^\infty - 5\left[\frac{e^{-st}}{s^2}\left[st + 1\right]\right]_0^\infty - \frac{3}{s}\left[e^{-st}\right]_0^\infty$$

$$= \left[0 - \frac{2}{-s^3}\right] - 5\left[0 - \frac{1}{s^2}\right] - \frac{3}{s}[0 - 1] \quad = \frac{2}{s^3} + \frac{5}{s^2} + \frac{3}{s} \quad = \frac{2 + 5s + 3s^2}{s^3}, \quad s > 0$$

Now that we have $F(s)$ we can finish finding the last term for the system equation:

$$Df(t) \Leftrightarrow sF(s) - f(0)$$

$$= s\,\frac{2+5s+3s^2}{s^3} - 3 \qquad = \frac{2+5s+3s^2}{s^2} - 3$$

$$= \frac{2+5s+3s^2-3s^2}{s^2} \qquad = \frac{2+5s}{s^2}, \qquad s > 0$$

Substituting the Laplace terms into the system equation we have:

$$D^2 y(t) + 3Dy(t) + 2y(t) = Df(t) \qquad \text{[system equation]}$$

$$\left[s^2 Y(s) - 2s - 3\right] + \left[3sY(s) - 6\right] + \left[2Y(s)\right] = \frac{2+5s}{s^2}, \qquad s > 0$$

Collecting like terms we have:

$$(s^2+3s+2)Y(s) - 2s - 3 - 6 = \frac{2+5s}{s^2}, \qquad s > 0$$

$$(s^2+3s+2)Y(s) = \frac{2+5s}{s^2} + 2s + 9$$

$$(s+1)(s+2)Y(s) = \frac{2s^3+9s^2+5s+2}{s^2}$$

$$Y(s) = \frac{2s^3+9s^2+5s+2}{s^2(s+1)(s+2)}$$

Applying partial fractions:

$$Y(s) = \frac{2s^3+9s^2+5s+2}{s^2(s+1)(s+2)} \qquad = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1} + \frac{D}{s+2}$$

$$A = \frac{2}{1\times 2} = 1 \qquad B = \frac{2}{1\times 2} = 1$$

$$C = \frac{2(-1)^3+9(-1)^2+5(-1)+2}{(-1)^2(1)} = \frac{-2+9-5+2}{1} = 4$$

$$D = \frac{2(-2)^3+9(-2)^2+5(-2)+2}{(-2)^2(-1)} - \frac{-16+36-10+2}{-4} = -3$$

$$Y(s) = \frac{1}{s^2} + \frac{1}{s} + \frac{4}{s+1} - \frac{3}{s+2}$$

From the table of Laplace transforms, we know that:

$$u(t) \Leftrightarrow \frac{1}{s}, \quad tu(t) \Leftrightarrow \frac{1}{s^2}, \quad e^{\lambda t}u(t) \Leftrightarrow \frac{1}{s-\lambda}$$

so we have:

$$y(t) = tu(t) + u(t) + 4e^{-t}u(t) - 3e^{-2t}u(t)$$

$$y(t) = \left(4e^{-t} - 3e^{-2t} + t + 1\right)u(t)$$

THE ANSWER

Therefore we have:

$$\boxed{y(t) = 4e^{-t} - 3e^{-2t} + t + 1, \quad t > 0}$$