THE FOURIER TRANSFORM

Fundamentals of the Fourier Transform

The Fourier transform is the **frequency-domain** representation of a function. The Fourier transform is actually a special case of the Laplace transform in which $s = j\omega$. That is, in the case of the Fourier transform "s" is restricted to the imaginary axis on the complex frequency plane. Consequently, use of the Fourier transform is limited to asymptotically stable systems and only if the inputs are Fourier-transformable. The Fourier transform is preferred for **signal** analysis and the Laplace transform is preferred for **system** analysis.

THE FOURIER TRANSFORM

The Fourier transform of a function $f(t)$ is expressed symbolically as $F(\omega)$, (though our professor preferred to use $F(j\omega)$ for the same thing), where ω is frequency in rad/s.

$$\mathcal{F}[f(t)] = F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

THE INVERSE FOURIER TRANSFORM

$$\mathcal{F}^{-1}[f(\omega)] = F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)e^{j\omega t}\,d\omega$$

FREQUENCY SPECTRA

AMPLITUDE SPECTRUM - The plot of amplitude $|F(\omega)|$ versus the (radian) frequency ω.

$$|F(\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}$$

PHASE SPECTRUM - The plot of the phase $\angle F(\omega)$ versus the (radian) frequency ω.

$$\angle F(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

These two plots convey all of the information that the plot of $f(t)$ as a function of t does. The frequency spectra of a signal constitute the **frequency-domain** description of $f(t)$ whereas $f(t)$ as a function of t is the **time-domain** description.

A TABLE OF FOURIER TRANSFORMS

	$f(t)$	$F(w)$	
1	$e^{-at}u(t)$	$\dfrac{1}{a+j\omega}$,	$a>0$
2	$e^{at}u(-t)$	$\dfrac{1}{a-j\omega}$,	$a>0$
3	$e^{-a\|t\|}$	$\dfrac{2a}{a^2+\omega^2}$,	$a>0$
4	$te^{-at}u(t)$	$\dfrac{1}{(a+j\omega)^2}$,	$a>0$
5	$t^n e^{-at}u(t)$	$\dfrac{n!}{(a+j\omega)^{n+1}}$,	$a>0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	
9	$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	
10	$\sin\omega_0 t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$	
11	$u(t)$	$\pi\delta(\omega)+\dfrac{1}{j\omega}$	
12	$\operatorname{sgn} t$	$\dfrac{2}{j\omega}$	
13	$\cos\omega_0 t\, u(t)$	$\dfrac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\dfrac{j\omega}{\omega_0^{\ 2}-\omega^2}$	
14	$\sin\omega_0 t\, u(t)$	$\dfrac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\dfrac{\omega_0}{\omega_0^{\ 2}-\omega^2}$	
15	$e^{-at}\sin\omega_0 t\, u(t)$	$\dfrac{\omega_0}{(a+j\omega)^2+\omega_0^{\ 2}}$,	$a>0$
16	$e^{-at}\cos\omega_0 t\, u(t)$	$\dfrac{a+j\omega}{(a+j\omega)^2+\omega_0^{\ 2}}$,	$a>0$

A TABLE OF FOURIER TRANSFORM OPERATIONS

Operation	***f*(*t*)**	***F*(w)**
Addition	$f_1(t) + f_2(t)$	$F_1(\omega) + F_2(\omega)$
Scalar multiplication	$kf(t)$	$kF(\omega)$
Duality	$F(t)$	$2\pi f(-\omega)$
Scaling	$f(at)$	$\frac{1}{\lvert a\rvert} F\left(\frac{\omega}{a}\right)$
Time shift	$f(t - t_0)$	$F(\omega)e^{-j\omega t_0}$
Frequency shift	$f()e^{j\omega_0 t}$	$F(\omega - \omega_0)$
Time convolution	$f_1(t) * f_2(t)$	$F_1(\omega)\, F_2(\omega)$
Frequency convolution	$f_1(t)\ f_2(t)$	$\frac{1}{2\pi} F_1(\omega) * F_2(\omega)$
Time differentiation	$\frac{d^n f}{dt^n}$	$(j\omega)^n F(\omega)$
Time integration	$\int_{-\infty}^{t} f(x)\, dx$	$\frac{F(\omega)}{j\omega} + \pi F(0)\,\delta(\omega)$