## VARIATION OF PARAMETERS

Method for solving a non-homogeneous second order differential equation This method is more difficult than the method of undetermined coefficients but is useful in solving more types of equations such as this one with repeated roots.
49. p185 $y^{\prime \prime}-4 y^{\prime}+4 y=2 e^{2 x}$ Find a particular solution.

The characteristic equation of the associated homogeneous equation is: $r^{2}-4 r+4=0$
which factors to $(r-2)(r-2)=0$ having repeated roots $r=2$.
so that the general solution of the associated homogeneous equation is $y_{c}=c_{1} e^{2 x}+c_{2} x e^{2 x}$
From this equation we determine our $y_{1}$ and $y_{2}$ values to be $y_{1}=e^{2 x}$ and $y_{2}=x e^{2 x}$ from which we get $y_{1}{ }^{\prime}=2 e^{2 x}$ and $y_{2}{ }^{\prime}=2 x e^{2 x}+e^{2 x}$

| From the textbook we have | $u_{1}{ }^{\prime} y_{1}+u_{2}{ }^{\prime} y_{2}=0$ and | $u_{1}{ }^{\prime} y_{1}{ }^{\prime}+u_{2}{ }^{\prime} y_{2}{ }^{\prime}=f(x)$ |
| :--- | :--- | :--- |
| into which we may substitute | $u_{1}{ }^{\prime} e^{2 x}+u_{2}{ }^{\prime} x e^{2 x}=0$ | $u_{1}{ }^{\prime} 2 e^{2 x}+u_{2}{ }^{\prime}\left(2 x e^{2 x}+e^{2 x}\right)=2 e^{2 x}$ |
| reduce and solve simultaneously | $u_{1}{ }^{\prime}+x u_{2}{ }^{\prime}=0$ | $2 u_{1}{ }^{\prime}+2 x u_{2}{ }^{\prime}+u_{2}{ }^{\prime}=2$ |
|  | $u_{1}{ }^{\prime}=-x u_{2}{ }^{\prime}$ | $2\left(-x u_{2}{ }^{\prime}\right)+2 x u_{2}{ }^{\prime}+u_{2}{ }^{\prime}=2$ |
|  | $u_{1}{ }^{\prime}=-2 x$ | $u_{2}{ }^{\prime}=2$ |

Integrate to find $u_{1}$ and $u_{2}$

$$
u_{1}=-2 \int x d x=-x^{2} \quad u_{2}=2 \int d x=2 x
$$

Using the formula for a particular solution $y_{p}(x)=u_{1} y_{1}+u_{2} y_{2}$ we have $y_{p}(x)=-x^{2} e^{2 x}+2 x\left(x e^{2 x}\right)$ which simplifies to the expression $y_{p}(x)=x^{2} e^{2 x}$

We then append this term to the general solution for the associated homogeneous equation to obtain the general solution

$$
y_{c}=c_{1} e^{2 x}+c_{2} x e^{2 x}+x^{2} e^{2 x}
$$

If initial values are given, they can be applied to this solution and its derivative.

