## VARIATION OF PARAMETERS

**Method for solving a non-homogeneous second order differential equation** This method is more difficult than the method of undetermined coefficients but is useful in solving more types of equations such as this one with repeated roots.

**49.** p185  $y'' - 4y' + 4y = 2e^{2x}$  Find a particular solution.

The characteristic equation of the associated homogeneous equation is:  $r^2 - 4r + 4 = 0$ which factors to (r-2)(r-2) = 0 having repeated roots r = 2. so that the general solution of the associated homogeneous equation is  $y_c = c_1 e^{2x} + c_2 x e^{2x}$ 

From this equation we determine our  $y_1$  and  $y_2$  values to be  $y_1 = e^{2x}$  and  $y_2 = xe^{2x}$ from which we get  $y_1' = 2e^{2x}$  and  $y_2' = 2xe^{2x} + e^{2x}$ 

From the textbook we have into which we may substitute reduce and solve simultaneously

$$u_1' y_1 + u_2' y_2 = 0$$
 and
  $u_1' y_1' + u_2' y_2' = f(x)$ 
 $u_1' e^{2x} + u_2' xe^{2x} = 0$ 
 $u_1' 2e^{2x} + u_2' (2xe^{2x} + e^{2x}) = 2e^{2x}$ 
 $u_1' + xu_2' = 0$ 
 $u_1' 2e^{2x} + u_2' (2xe^{2x} + e^{2x}) = 2e^{2x}$ 
 $u_1' + xu_2' = 0$ 
 $2u_1' + 2xu_2' + u_2' = 2$ 
 $u_1' = -xu_2'$ 
 $2(-xu_2') + 2xu_2' + u_2' = 2$ 
 $u_1' = -2x$ 
 $u_2' = 2$ 
 $u_1 = -2\int x dx = -x^2$ 
 $u_2 = 2\int dx = 2x$ 

Integrate to find  $u_1$  and  $u_2$ 

Using the formula for a particular solution  $y_p(x) = u_1 y_1 + u_2 y_2$ we have  $y_p(x) = -x^2 e^{2x} + 2x(xe^{2x})$  which simplifies to the expression  $y_p(x) = x^2 e^{2x}$ 

We then append this term to the general solution for the associated homogeneous equation to obtain the general solution

$$y_c = c_1 e^{2x} + c_2 x e^{2x} + x^2 e^{2x}$$

If initial values are given, they can be applied to this solution and its derivative.