POWER SERIES

Solving a second order homogeneous differential equation using the power series

17. p479
$$y'' + xy' - 2y = 0$$
 $y(0) = 1, y'(0) = 0$

To solve this problem we will:

- 1. Set *y* equal to a power series and substitute the power series and its derivatives into the equation for *y* and the derivatives of *y*.
 - 2. Manipulate the expression so that *x* and the summation operators can be eliminated.
 - 3. Form a *recurrence relation* from this expression by solving for a constant other than c_n .
 - 4. Substitute values for *n* into the *recurrence relation* to form two new series expressions involving two independent constants. This is the general solution in rough form.
 - 5. Simplify these series as sums where possible by recognizing patterns which can be expressed as sums.
 - 6. Substitute the first initial value into the solution and substitute the second initial value into its derivative to determine the values of the two constants.
 - 7. Substitute the values of the two constants into the general solution to find the particular solution.

let
$$y = \sum_{n=0}^{\infty} c_n x^n$$
 then $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$

Substitute these expressions into the original equation

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x \sum_{n=1}^{\infty} nc_n x^{n-1} - 2\sum_{n=0}^{\infty} c_n x^n = 0$$

We want to factor out the x term from each expression, so we must manipulate them so that each x is raised to the same power n and each series is evaluated starting from the same initial value of n.

In the first series an *n*-shift is performed by replacing *n* with *n*+2, resulting in $\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$

In the second series we merely carry out the multiplication by x so that $y' = \sum_{n=1}^{\infty} nc_n x^n$

Before we can begin factoring we still need all of our series to have the same initial value *n*. One way this can be done is to remove the offending terms. We could remove the *n*=0 terms from the first and last series and set them equal to zero, i.e. $(0+2)(0+1)c_{0+2}x^0 - 2c_0x^0 = 0$ We could solve for these values and then solve for the remainder by rewriting the sum of series with each defined initially at *n*=1. However in this particular problem there is an easier way. Since the *n*=0 term of the second sum would be equal to

zero, it would have no effect on the equation if we simply change the n=1 to n=0, so that we are now evaluating each of the three series from the same starting point, zero. So now we have

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n + \sum_{n=0}^{\infty} nc_n x^n - 2\sum_{n=0}^{\infty} c_n x^n = 0$$

By the *Identity Principle* we can factor out the x^n and the summation operators to get $(n+1)(n+2)c_{n+2} + (n-2)c_n = 0$

Solve for c_{n+2} to obtain the *recurrence relation* $c_{n+2} = \frac{(2-n)c_n}{(n+1)(n+2)}$

If we were not so fortunate to have c_n in the term on the right, but instead had say c_{n-1} it would be necessary to do another *n*-shift to realize the c_n term.

Next we plug in even and odd values for n (In some situations the resulting series might not be found by the even and odd values but say every third value.)

n = 0:	$c_2 = \frac{2c_0}{2}$				
<i>n</i> = 2:	$c_4 = 0$ All further even values of <i>n</i> will result in zero.				
<i>n</i> = 1:	$c_3 = \frac{c_1}{2 \cdot 3}$ Do not simplify the results.				
<i>n</i> = 3:	$c_5 = \frac{-c_3}{4 \cdot 5} = \frac{-c_1}{2 \cdot 3 \cdot 4 \cdot 5}$				
<i>n</i> = 5:	$c_7 = \frac{-3c_5}{6 \cdot 7} = \frac{3c_1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}$ A pattern emerges in the denominator				
<i>n</i> = 7:	$c_9 = \frac{-5c_7}{8 \cdot 9} = \frac{3 \cdot 5c_1}{9!}$				
<i>n</i> = 9:	$c_{11} = \frac{-7c_9}{10 \cdot 11} = \frac{3 \cdot 5 \cdot 7c_1}{11!}$ A pattern emerges in the numerator though it is not consistent from the beginning of this series.				

So far, our general solution is

$$y(x) = c_0 + c_0 x^2 + c_1 x + \frac{c_1 x^3}{3!} - \frac{c_1 x^5}{5!} + \frac{3c_1 x^7}{7!} - \frac{3 \cdot 5c_1 x^9}{9!} + \frac{3 \cdot 5 \cdot 7c_1 x^{11}}{11!} \cdots$$

factoring out the constants $y(x) = c_0(1+x^2) + c_1\left(x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{3x^7}{7!} - \frac{3 \cdot 5x^9}{9!} + \frac{3 \cdot 5 \cdot 7x^{11}}{11!} \cdots\right)$

To see how to write the c_1 components in terms of a sum it will help to label the terms with the <u>new</u> *n* index.

<i>n</i> =	0	1	2	3	4	5
terms:	x	$\frac{x^3}{3!}$	$-\frac{x^5}{5!}$	$\frac{3x^7}{7!}$	$-\frac{3\cdot 5x^9}{9!}$	$\frac{3\cdot 5\cdot 7x^{11}}{11!}$

We can now observe that for the higher terms we have:

$$\sum \frac{(-1)^{n+1}(2n-3)!!x^{2n+1}}{(2n+1)!}$$

The $(-1)^{n+1}$ term gives the alternating sign of the series. The double factoral means only the odd terms are multiplied, i.e. $7!! = 7 \cdot 5 \cdot 3 \cdot 1$. Note the frequent appearance of 2n which comes from having originally derived the terms of this series from every other *n*.

However, this sum does not hold for the first two terms, so they will have to be written separately. The general solution is

$$y(x) = c_0(1+x^2) + c_1\left(x + \frac{x^3}{3!} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}(2n-3)!!x^{2n+1}}{(2n+1)!}\right)$$

Plugging in our first initial condition y(0) = 1 we have $1 = c_0(1+0) + c_1\left(0 + \sum_{n=2}^{\infty} 0\right)$ so that $c_0 = 1$

Taking the derivative of the general solution we have

$$y'(x) = c_0(2x) + c_1 \left(1 + \frac{1}{2}x^2 + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}(2n-3)!!(2n-1)x^{2n}}{(2n+1)!} \right)$$

Plugging in our second initial condition y'(0) = 0 we have $0 = c_0(0) + c_1 \left(1 + 0 + \sum_{n=2}^{\infty} 0\right)$ so that

 $c_1 = 0$

So the particular solution is	$y(x) = 1 + x^2$

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