## POWER SERIES

Solving a second order homogeneous differential equation using the power series
17. p479 $y^{\prime \prime}+x y^{\prime}-2 y=0 \quad y(0)=1, \quad y^{\prime}(0)=0$

To solve this problem we will:

1. Set $y$ equal to a power series and substitute the power series and its derivatives into the equation for $y$ and the derivatives of $y$.
2. Manipulate the expression so that $x$ and the summation operators can be eliminated.
3. Form a recurrence relation from this expression by solving for a constant other than $c_{n}$.
4. Substitute values for $n$ into the recurrence relation to form two new series expressions involving two independent constants. This is the general solution in rough form.
5. Simplify these series as sums where possible by recognizing patterns which can be expressed as sums.
6. Substitute the first initial value into the solution and substitute the second initial value into its derivative to determine the values of the two constants.
7. Substitute the values of the two constants into the general solution to find the particular solution.
let $y=\sum_{n=0}^{\infty} c_{n} x^{n} \quad$ then $y^{\prime}=\sum_{n=1}^{\infty} n c_{n} x^{n-1} \quad$ and $y^{\prime \prime}=\sum_{n=2}^{\infty} n(n-1) c_{n} x^{n-2}$
Substitute these expressions into the original equation

$$
\sum_{n=2}^{\infty} n(n-1) c_{n} x^{n-2}+x \sum_{n=1}^{\infty} n c_{n} x^{n-1}-2 \sum_{n=0}^{\infty} c_{n} x^{n}=0
$$

We want to factor out the $x$ term from each expression, so we must manipulate them so that each $x$ is raised to the same power $n$ and each series is evaluated starting from the same initial value of $n$.

In the first series an $n$-shift is performed by replacing $n$ with $n+2$, resulting in $\sum_{n=0}^{\infty}(n+2)(n+1) c_{n+2} x^{n}$
In the second series we merely carry out the multiplication by $x$ so that $y^{\prime}=\sum_{n=1}^{\infty} n c_{n} x^{n}$
Before we can begin factoring we still need all of our series to have the same initial value $n$. One way this can be done is to remove the offending terms. We could remove the $n=0$ terms from the first and last series and set them equal to zero, i.e. $(0+2)(0+1) c_{0+2} x^{0}-2 c_{0} x^{0}=0 \quad$ We could solve for these values and then solve for the remainder by rewriting the sum of series with each defined initially at $n=1$. However in this particular problem there is an easier way. Since the $n=0$ term of the second sum would be equal to
zero, it would have no effect on the equation if we simply change the $n=1$ to $n=0$, so that we are now evaluating each of the three series from the same starting point, zero. So now we have

$$
\sum_{n=0}^{\infty}(n+2)(n+1) c_{n+2} x^{n}+\sum_{n=0}^{\infty} n c_{n} x^{n}-2 \sum_{n=0}^{\infty} c_{n} x^{n}=0
$$

By the Identity Principle we can factor out the $x^{n}$ and the summation operators to get $(n+1)(n+2) c_{n+2}+(n-2) c_{n}=0$

Solve for $c_{n+2}$ to obtain the recurrence relation $c_{n+2}=\frac{(2-n) c_{n}}{(n+1)(n+2)}$
If we were not so fortunate to have $c_{n}$ in the term on the right, but instead had say $c_{n-1}$ it would be necessary to do another $n$-shift to realize the $c_{n}$ term.

Next we plug in even and odd values for $n$ (In some situations the resulting series might not be found by the even and odd values but say every third value.)

| $n=0:$ | $c_{2}=\frac{2 c_{0}}{2}$ |
| :--- | :--- |
| $n=2:$ | $c_{4}=0 \quad$ All further even values of $n$ will result in zero. |


| $n=1:$ | $c_{3}=\frac{c_{1}}{2 \cdot 3} \quad$ Do not simplify the results. |
| :--- | :--- |
| $n=3:$ | $c_{5}=\frac{-c_{3}}{4 \cdot 5}=\frac{-c_{1}}{2 \cdot 3 \cdot 4 \cdot 5}$ |
| $n=5:$ | $c_{7}=\frac{-3 c_{5}}{6 \cdot 7}=\frac{3 c_{1}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} \quad$ A pattern emerges in the denominator |
| $n=7:$ | $c_{9}=\frac{-5 c_{7}}{8 \cdot 9}=\frac{3 \cdot 5 c_{1}}{9!}$ |
| $n=9:$ | $c_{11}=\frac{-7 c_{9}}{10 \cdot 11}=\frac{3 \cdot 5 \cdot 7 c_{1}}{11!} \quad$A pattern emerges in the numerator though it is not <br> consistent from the beginning of this series. |

So far, our general solution is

$$
y(x)=c_{0}+c_{0} x^{2}+c_{1} x+\frac{c_{1} x^{3}}{3!}-\frac{c_{1} x^{5}}{5!}+\frac{3 c_{1} x^{7}}{7!}-\frac{3 \cdot 5 c_{1} x^{9}}{9!}+\frac{3 \cdot 5 \cdot 7 c_{1} x^{11}}{11!} \cdots
$$

factoring out the constants $y(x)=c_{0}\left(1+x^{2}\right)+c_{1}\left(x+\frac{x^{3}}{3!}-\frac{x^{5}}{5!}+\frac{3 x^{7}}{7!}-\frac{3 \cdot 5 x^{9}}{9!}+\frac{3 \cdot 5 \cdot 7 x^{11}}{11!} \cdots\right)$

To see how to write the $c_{1}$ components in terms of a sum it will help to label the terms with the new $n$ index.

| $n=$ | 0 | 1 | 2 | 3 | 4 | 5 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| terms: | $x$ | $\frac{x^{3}}{3!}$ | $-\frac{x^{5}}{5!}$ | $\frac{3 x^{7}}{7!}$ | $-\frac{3 \cdot 5 x^{9}}{9!}$ | $\frac{3 \cdot 5 \cdot 7 x^{11}}{11!}$ |

We can now observe that for the higher terms we have:

$$
\sum \frac{(-1)^{n+1}(2 n-3)!!x^{2 n+1}}{(2 n+1)!}
$$

The $(-1)^{n+1}$ term gives the alternating sign of the series. The double factoral means only the odd terms are multiplied, i.e. $7!!=7 \cdot 5 \cdot 3 \cdot 1$. Note the frequent appearance of $2 n$ which comes from having originally derived the terms of this series from every other $n$.

However, this sum does not hold for the first two terms, so they will have to be written separately. The general solution is

$$
y(x)=c_{0}\left(1+x^{2}\right)+c_{1}\left(x+\frac{x^{3}}{3!}+\sum_{n=2}^{\infty} \frac{(-1)^{n+1}(2 n-3)!!x^{2 n+1}}{(2 n+1)!}\right)
$$

Plugging in our first initial condition $y(0)=1$ we have $1=c_{0}(1+0)+c_{1}\left(0+\sum_{n=2}^{\infty} 0\right)$ so that $c_{0}=1$
Taking the derivative of the general solution we have

$$
y^{\prime}(x)=c_{0}(2 x)+c_{1}\left(1+\frac{1}{2} x^{2}+\sum_{n=2}^{\infty} \frac{(-1)^{n+1}(2 n-3)!!(2 n-1) x^{2 n}}{(2 n+1)!}\right)
$$

Plugging in our second initial condition $y^{\prime}(0)=0$ we have $0=c_{0}(0)+c_{1}\left(1+0+\sum_{n=2}^{\infty} 0\right)$ so that $c_{1}=0$

So the particular solution is $y(x)=1+x^{2}$

